Exploring Mathematical Functions through Dynamic Microworlds

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Abstract

The aim of this research was to investigate students' perceptions of function as they interacted with the different dynamic representations of function made available through computer environments. Microworlds were designed comprising sequences of activities around the software, Function Probe, and two adaptations of DynaGraph, DG Parallel (with parallel axes) and DG Cartesian (using Cartesian axes). A series of case studies of four pairs of students was undertaken in Brazil in order to trace the evolution in students' perceptions of a selection of function properties; namely turning point, variation, range, symmetry and periodicity. This diversity of properties was chosen to examine different ways students analyse functions: pointwise, variational, global and pictorial.

Starting with an examination of the curriculum followed by the case study students as a means to describe the origins of their perceptions, a longitudinal investigation was undertaken in order to identify the main features of each of the microworlds that appeared to contribute to students' progress. The students' perceptions were analysed by drawing attention to their origins, their usefulness and their potential limitations (from a mathematical point of view). A methodology for this longitudinal study was devised which incorporated visual presentations to capture the main characteristics of students' perceptions.

The results showed that DG Parallel, a 'new' representation, prompted the development of perceptions free of previous limitations and sufficiently robust to allow revision. However, properties previously perceived pictorially were rarely identified in DG Parallel. Together with DG Cartesian, interactions with this microworld provoked the students to develop a variational view of some of the function properties. In addition, DG Cartesian served as a two-way bridge between variational and pictorial views. By way of contrast, using the tools in FP to transform graphs seemed not to shape perceptions, but to assist in the exploration of the function properties.

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Acknowledgements

I would like to express my gratitude to my supervisor, Prof. Celia Hoyles, for her central role in the whole process of elaboration of this study. All her insightful comments on my work contributed enormously to my formation as a researcher. I also thank Prof. Alison Wolf and Prof. Richard Noss for their valuable suggestions as members of my annual Review Board. My friend, Elizabeth Moren, helped me considerably by reading and discussing the earlier drafts of many chapters of this thesis.

This degree was made possible by funding from CAPES — a Brazilian institution for supporting academic development. CAPES' members also did all they could to help me at all times. In fact, I am grateful to the people of Brazil, who ultimately fund CAPES. The Federal University of Pernambuco in Brazil also invested in my academic career by giving me leave of absence from work.

The friendly teacher and students who participated in the empirical study were so generous and important to the development of this study. The school where I did the empirical study also allowed me to access the necessary information

Paulo Figueiredo, Licia Maia, Paulo Gileno, Sandra Magina and Maria Antonia MacDowell were some of my academic colleagues and professors in Brazil who, by trusting in my potential, encouraged me to start, continue and finish this research.

All thanks to the staff of Mathematical Science Group, especially Wendy Robins, Alison Shapton and Dorothy Pattman in the office, who during these four years gave me friendly help. In particular, Dorothy Pattman patiently corrected the English of this thesis.

Personally, I am grateful to all my relatives, especially to my husband, Joaquim Tavares, one of my brothers, Mauricio Antonio Gitirana Gomes Ferreira, and my parents Augusto Burle Gomes Ferreira and Eliane Gitirana Gomes Ferreira, who spared no effort in encouraging me to undertake this degree. They were the ones who many times turned from their own concerns to give me emotional support. I also thank the friends I have made in this country whose co-operative spirit has given me vital support in developing this study.

Chapters

I — The Study

1 Aims

This research investigates students' perceptions of function as they interact with different dynamic representations of function available through computer environments.

A selection of properties of function are distinguished and the study seeks to analyse how students come to discriminate, generalise, and synthesise these properties while working with chosen software programs in activities designed to encourage exploration of the dynamic features of the programs. The software used will be: DynaGraph (Goldenberg et al, 1992) and Function Probe (FP) (Confrey et al, 1991a). Two adaptations of DynaGraph will be implemented: one parallel version (DG Parallel) and one Cartesian version (DG Cartesian). The set of activities around each software will be described as a microworld.

The research focuses on the following set of aims:

- an analysis of students' perceptions of the following properties of mathematical function: *range, periodicity, variation, turning point* and *symmetry*;
- the identification of differences and similarities in students' perceptions of these properties during interaction in the different microworlds;
- the identification of any sources of difficulty;
- the tracing of trajectories of learning;
- the identification, where possible, of the antecedents of any difficulty particularly in so far as this might originate in the Brazilian curricula;
- the identification of how links come to be forged between the different perceptions of a property of function as evidenced in the different microworlds and between these perceptions and students' previous knowledge.

2 The underlying rationale

I start from the position that different representations have different influences on students' perceptions of the properties of function. Different representations emphasise different aspects of the same concept; one representation can facilitate students' perceptions of one property, while making it harder for them to perceive other properties. Following a similar argument, the main assumption of this thesis is that dynamic visual tools available in FP and in DynaGraph will lead students to differentially emphasise the properties of function as well as to perceive them in a different light.

Additionally, this study assumes that by describing and comparing functions as represented in different microworlds, students will be provoked to revise and generalise their perceptions of the chosen properties of function.

3 Research questions

The study attempts to address the research question:

> How does interaction with the dynamic tools offered by DynaGraph and Function Probe structure students' perceptions of the following properties of function: range, turning point, symmetry, variation and periodicity?

The following questions adress the interaction in detail:

Q1: How do students discriminate and generalise these properties in each microworld?

Q2: How does their knowledge of school mathematics affect their perceptions of these properties?

Q3: What role do the dynamic software tools play in helping students to overcome obstacles and any limitations in their perceptions?

Q4: Are these different perceptions synthesised by the students? If so, how? If not, why not?

Q5: How do explorations of the dynamic tools of Function Probe and DynaGraph change students' previous knowledge?

4 The concept of function

The history of mathematics shows that the study of functions has been emphasised differently over time. Early studies on functionality together with the evolution of its definition reflect these changes in emphasis showing how functions were perceived. The concept of function has evolved from a geometric approach in the seventeenth century, through an algebraic approach in the eighteenth century to a set-theoretical approach in modern times.

Since pre-historic times civilisation has been interested in understanding the functional behaviour of natural processes (Boyer, 1946) such as the relation between the phases of the moon and the days of a month. In medieval mathematics, without any abstractions or definitions of the concept, functionality was studied as the science of dynamics. Rates of change such as speed and acceleration were the focus of these discussions. Even later, when the term was first used, the study of functions reflected the preoccupation with describing how variation in one quantity can affect variation in another — a variational view. According to Malik (1980), in the 17th century "The investigation of a relation between two *varying quantities* [my emphasis] had been fundamental in arriving at the concept of function" (p.490).

The first appearance of the term 'function' was in 1692 with Leibniz and Bernoulli, who adopted it "to designate certain variable geometrical quantities — such as ordinate, tangents, and radii of curvature - connected with given curves" (Boyer, 1946: 12). On being linked with curves, the term received a geometrical approach which involved also a variational view.

In the 18th century, mathematicians developed another definition which treated the concept of function essentially as an equation. For them a function was: "an analytic expression representing the relation between two variables with its graph having no corners" (Malik, 1980: 490). As pointed out by Boyer (1946), "The word function, as introduced by Leibnitz and as used during the eighteenth century, was essentially equivalent to the word formula" (p.12).

Despite this new definition, the geometric approach of function was not lost. "Euler saw that any curve drawn free hand in a plane determines a functional relationship which may not be representable, either implicitly or explicitly, in ordinary analytical form" (Boyer, 1946: 12). This observation was used by Lacroix to give a broader scope for the term function. For him, "Any quantity the value of which depends on one or more other quantities is said to be a function of the latter, whether or not one knows by what operations one can pass from the latter to the first quantity" (op.cit.: 12-13). Nonetheless, by his illustrations, Lacroix showed that he was still considering functions given by formulae or equation. In 1837, Dirichlet revised the definition of function to: "y is a function of x, for a given domain of values of x, whenever a precise law of correspondence between x and y can be stated clearly" (op.cit.: 13) where he meant by 'precise law' a rule which gives to x one and only one value of y. He intended to include badly-behaved functions such as the well-known Dirichlet's totally discontinuous curves, which is given by $y=f(x)$ is 1 if x is rational or 0 otherwise. The unicity of a function was highlighted.

Malik (1980) points out that with the introduction of topology and metric spaces, mathematicians realised that the properties of a function depended very much on sets (domain and range). "In 1917, Caratheodory defined a function as a rule of correspondence between a set A to real numbers and in 1939 Bourbaki defined function as a rule corresponding to two sets and in later chapters observed that it is a subset of the Cartesian product of sets" (p.491). The Dirichlet-Bourbaki definition appeared as:

> 'A function f from A to B is defined as any subset of the Cartesian product of A and B, such that, for every $a \in A$ there is exactly one $b \in B$ such that $(a,b) \in f$'.

This definition is a set-theoretical approach to functions which emphasises the concept as a mathematical entity.

As Burn (1993) explains, in English education, there is a contrast between the way function is explored at university level and at high school level. He argues that at university, functions are treated as they were by mathematicians at the beginning of the 19th century while at school level functions are treated as in the 17th and 18th centuries, where notions such as limits and real numbers are not explored.

School mathematics following a traditional approach has introduced students to the concept of function using the Dirichlet-Bourbaki definition. In line with traditional school mathematics, the majority of Brazilian secondary schools present this definition in the following way:

> 'Given A and B two sets, a relationship f is said to be a function if and only if for every element $a \in A$ exists only one element $b \in B$ such that $f(a)=b$'.

Although these schools introduce functions in a set-theoretical approach, the examples explored in general consist of functions specified by their equations. As shown by Vinner & Dreyfus (1989) and argued by Malik (1980), students do not use the definition to build their perceptions. "A student retains a concept only if it is used in the course; if only its particular form is used, the student unconsciously accepts the particular form ..." (Malik, 1980: 490-491). The majority of students rarely perceive function as a mathematical entity. Analysing students' and teachers' perceptions and definition, Vinner & Dreyfus (1989) classified them in the following ways: as a correspondence, as a rule, as a dependence relation, as an operation, as a formula, or as appearance of function in a determined representation.

Considering both the evolution of the concept of function and the classification made by Vinner & Dreyfus (1989), I would like to discuss alternative ways of perceiving a function and analysing its properties. On perceiving a function as a correspondence or rule, students can adopt two views: variational and pointwise. In a variational view, a function is analysed by looking at 'how the change from x_1 to x_2 is related to the change from y_1 to y_2'. This view was emphasised in the first studies of functionality as well as in its geometrical approach. In a pointwise view, a function is analysed according to 'how x is associated to y'. A definition which takes the function as being almost defined by an equation seems to be closely related to this view. Also, as Malik (1980) argues, the Dirichlet-Bourbaki definition which is algebraic in its sense, "appeals to the discrete faculty of thinking and lacks a feel for the variable" (p.492).

Vinner & Dreyfus (1989) showed that students' perception of a function also depends on the form in which it is expressed. When presented as a graph, function is usually perceived as a well-behaved curve. Research on students' understanding of graphs has pointed out that students usually interpret properties of function in a graph by its shape as a static picture (Goldenberg, 1988), which has been called a pictorial view. When presented by an equation, a function is essentially perceived as a process of taking one input [x] and obtaining one output [y], which has been called a procedural view. In circumstance when the students do not see x as a variable, this emphasis can lead them to analyse functions as the correspondence of points — a pointwise perception.

The present research will not take one of these views as the best way of dealing with the concept of function but rather the intention is to try to analyse the perceptions of the students while exploring the properties of function and to examine how students' ideas of the properties develop while interacting in each microworld. However, I have to consider that for each microworld, the designers intend to lead students to at least one of the views as distinguished above. For example, Goldenberg et al (1992) with DynaGraph intended to give students an opportunity to change their views of function from pointwise to variational. As regards Function Probe, while using multiple representations in contextual problems Confrey (1992a) intended to lead students to a variational view. These intentions will be analysed in section 5 of chapter III of after the description of the software in chapter II.

5 Description of the thesis

The thesis has ten chapters and five appendices. Chapters I, II, III and IV define the study. The present chapter introduces the aims, questions and arguments of the research and discusses the mathematical concept of function. Chapter II will describe the software programs used in the investigations. Chapter III will review the literature on function aiming to develop a theoretical framework from which to interpret the data and findings of the empirical study, following which the research questions will be presented in detail. Chapter IV will describe the methodology of the empirical study comprising four case studies with pairs of students. Appendix I will present worksheets used in the empirical study and appendix II will present the activities designed for the study around each software environment. Appendix III will present the steps used in the analysis of the data.

Chapters V, VI, VII and VIII will discuss the results of the empirical study. Chapter V will describe the pilot study and its findings. An analysis of how Brazilian schools approach the topic of function will be presented in chapter VI. Chapter VII will analyse the evolution of each pair of students' perceptions of the chosen function properties. Appendix IV will present tables and diagrams of students' perceptions of each of the chosen function properties. Chapter VIII will summarise and synthesise the work of all the pairs of students by comparing the findings from chapters VI and VII. Appendix V will present tables with evidence of the findings discussed in chapter VIII.

Chapters IX and X will conclude the research. Chapter IX will discuss the research findings in relation to other studies on function. Finally, chapter X will discuss issues arising from this study in relation to the research questions affecting the teaching and learning of mathematical function and the place of function in the school curriculum.

For reasons of simplicity, sections, sub-sections, figures, diagrams and tables will be denoted section (sub-section, figure, diagram or table) CN-No. (AN-No.) to refer to section (sub-section, figure, diagram or table) No. in chapter N (or in appendix N), for example table AIII-2.4 refers to table 2.4 in appendix III. When referring to a table, diagram or figure in the same section or the same chapter only the number will be used.

II — The Software Programs

Before the review of the literature on functions, brief descriptions of Function Probe and DynaGraph will be presented. The descriptions will focus on the features of the software programs which will be explored in the present study. The reader who already knows both software programs will not find it necessary to read this chapter.

1 Description of Function Probe

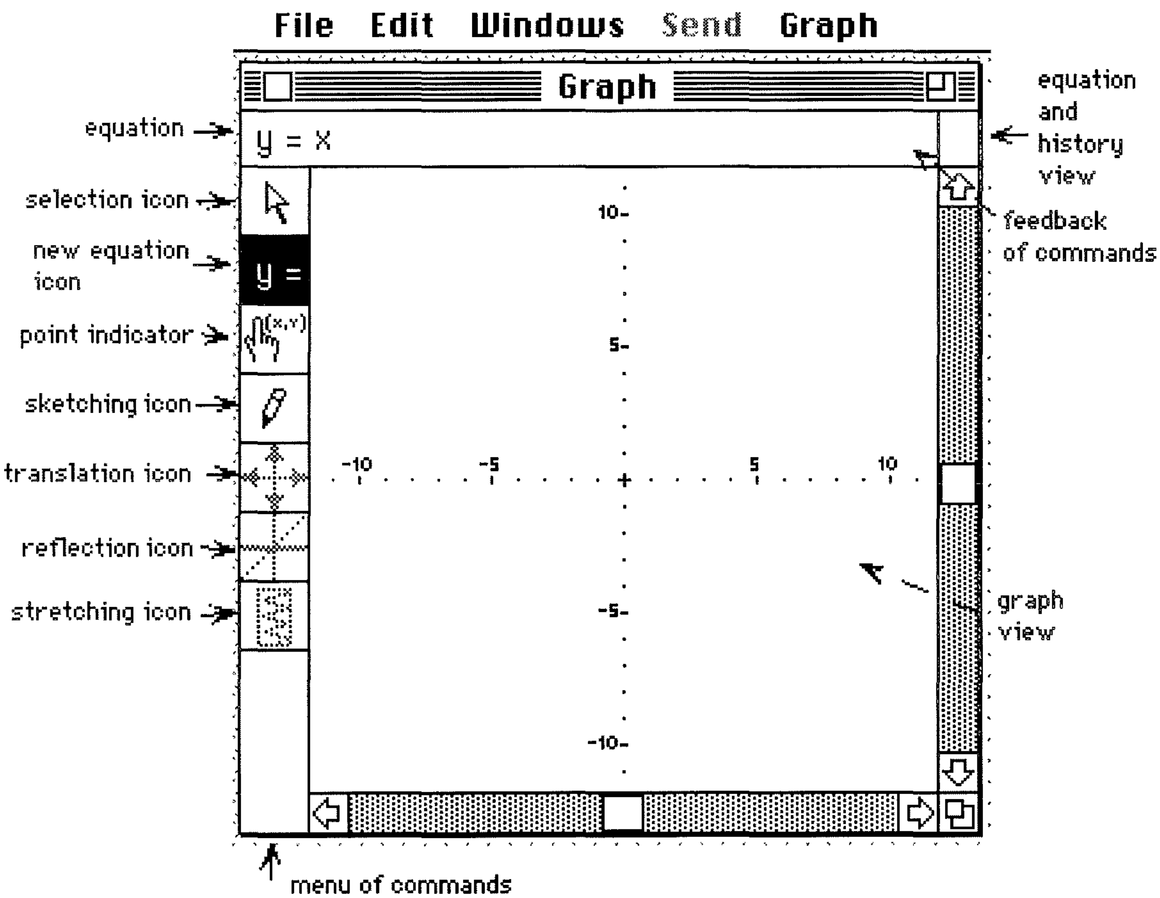
Function Probe (Confrey et al, 1991a) is a multiple representational software tool to enable students to explore the idea of functions. It combines three representations (equations, graphs and tables) in three windows (Graph, Table and Calculator). The integrity of each representation is preserved. Students can explore functions with actions either within one representation or with links made between different representations. This study will focus on the Graph window of FP particularly in the transformations students do in graphs while looking for properties of functions. Therefore, this section will present a description of the Graph window only. For a complete description of FP see Confrey et al (1991a). Also, section AI-4 presents a 'Journey through the software'.

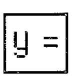
The Graph window presents both Cartesian and algebraic representations. This software allows dynamic transformations in graphs: stretching, translating and reflecting. Figure 1.1 presents the Graph window with FP menu.

The Graph window presents two spaces for the representations: the *equation and history view* and the *graph view*. The *graph view* presents a iconic *menu of commands*. Apart from the *new equation icon*, these commands, which include the transformations, are the actions allowed within graphs. *New equation icon* is one action between algebraic and Cartesian representations. A command can be selected by clicking the mouse on its icon. Apart from the *sketching icon*, the commands will be described below.

Figure 1.1

Graph window of Function Probe with menu





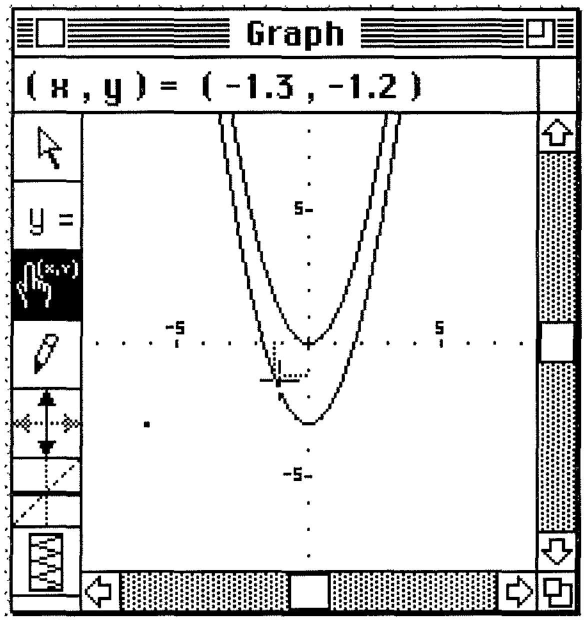
- *New equation icon* can be used to graph a function by its algebraic representation as input. Clicking the mouse on its icon, y= appears in the *equation* space in the *equation and history view*. Then, the user needs to write down the equation and press the [Return]-key. Function Probe traces its graph as feedback. Multiple graphs are allowed in the *graph view*.

- *Selection icon* is used to select one of the graphs presented on the *graph view*. When selected, the graph is highlighted in the *graph view* and its algebraic representation appears in the *equation* space, whenever possible. Selecting a graph is a necessary procedure to use the transformations: stretch, reflection and translation.

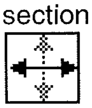
- *Point indicator* is used to plot points as well as to find out the coordinates of a point. As the icon is moved inside the *graph view*, the coordinates of the current point appears in the *equation* space. This command is particularly useful to localise points of a graph.

Figure 1.2

Point indicator being used



While exploring FP, this research will focus on the effects of the transformations of graphs on students' perceptions of function and of its properties. Therefore, I will give some examples of the execution of these transformations in graphs as a textual description of dynamic procedures is difficult. However, I really believe that the reader must try Function Probe at least once to grasp the real dimension of these transformations. Each of the transformations has at least two versions: vertical and horizontal. As the examples below only show the effects of these transformations, to learn how to operate them see the 'Journey through Function Probe software' in section AI-4.

 - When clicking the mouse on the *translation icon*, the user will be asked to select horizontal or vertical translation.

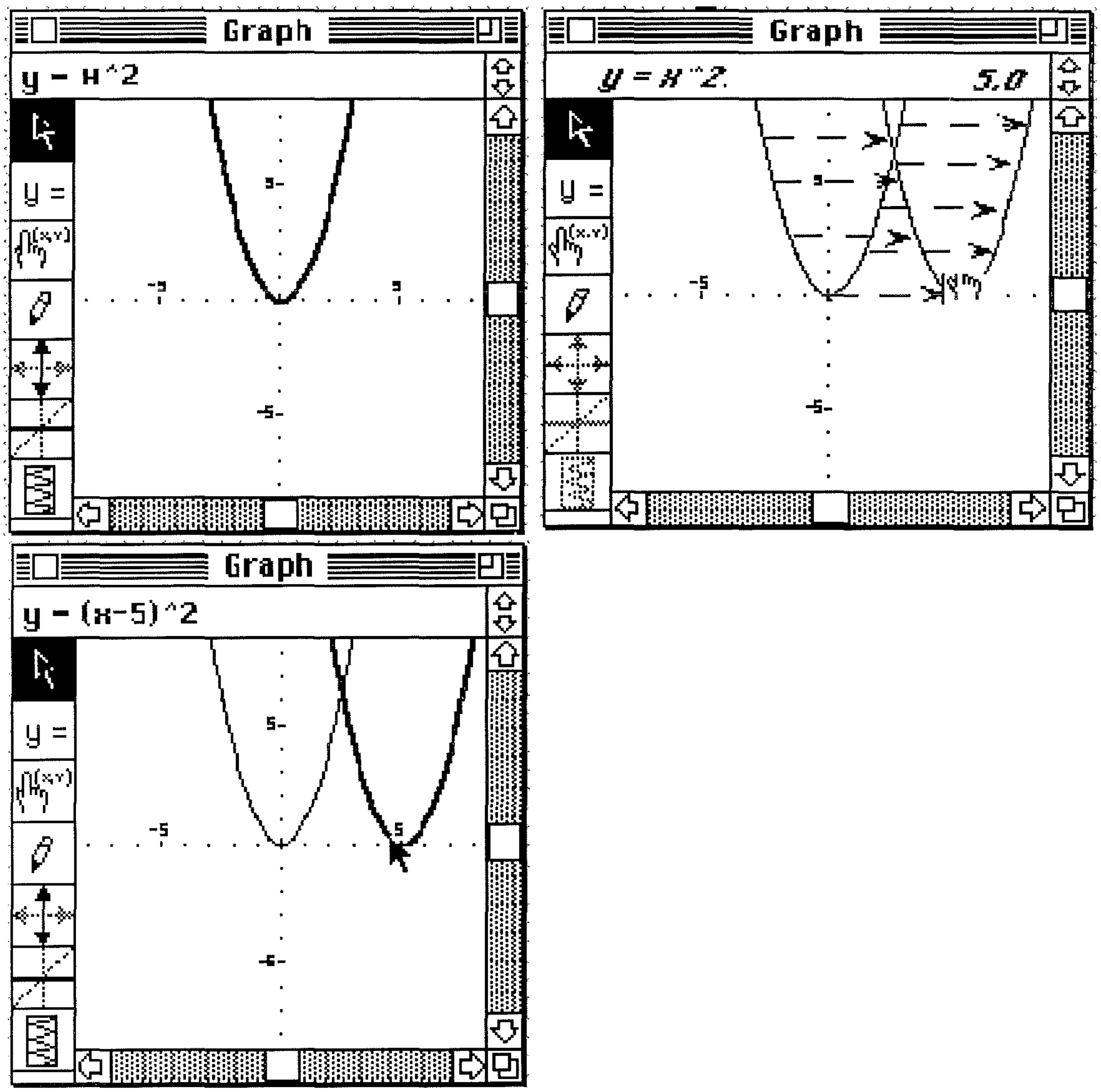
Horizontal translation is used to translate the graph in the direction of the x-axis, that is, horizontally. This is a dynamic process, i.e., student executes the translation seeing the intermediary phases of the transformation of the graph. Meanwhile, FP presents the number corresponding to the current transformation in the space for *feedback of the commands* at the right side of the *equation and history view*. Figure 1.3 shows a horizontal translation of +5 units in the graph of $y=x^2$. The second screen shows one intermediary phase of the transformation. Note that the equation modified appears in the *equation* space only when the transformation is finished.

Vertical translation is similar to *horizontal translation*, but translates the graph in the direction of the y-axis, that is, vertically.

Figure 1.3

Horizontal translation of the graph of $y=x^2$, in three phases:

before, during and after it

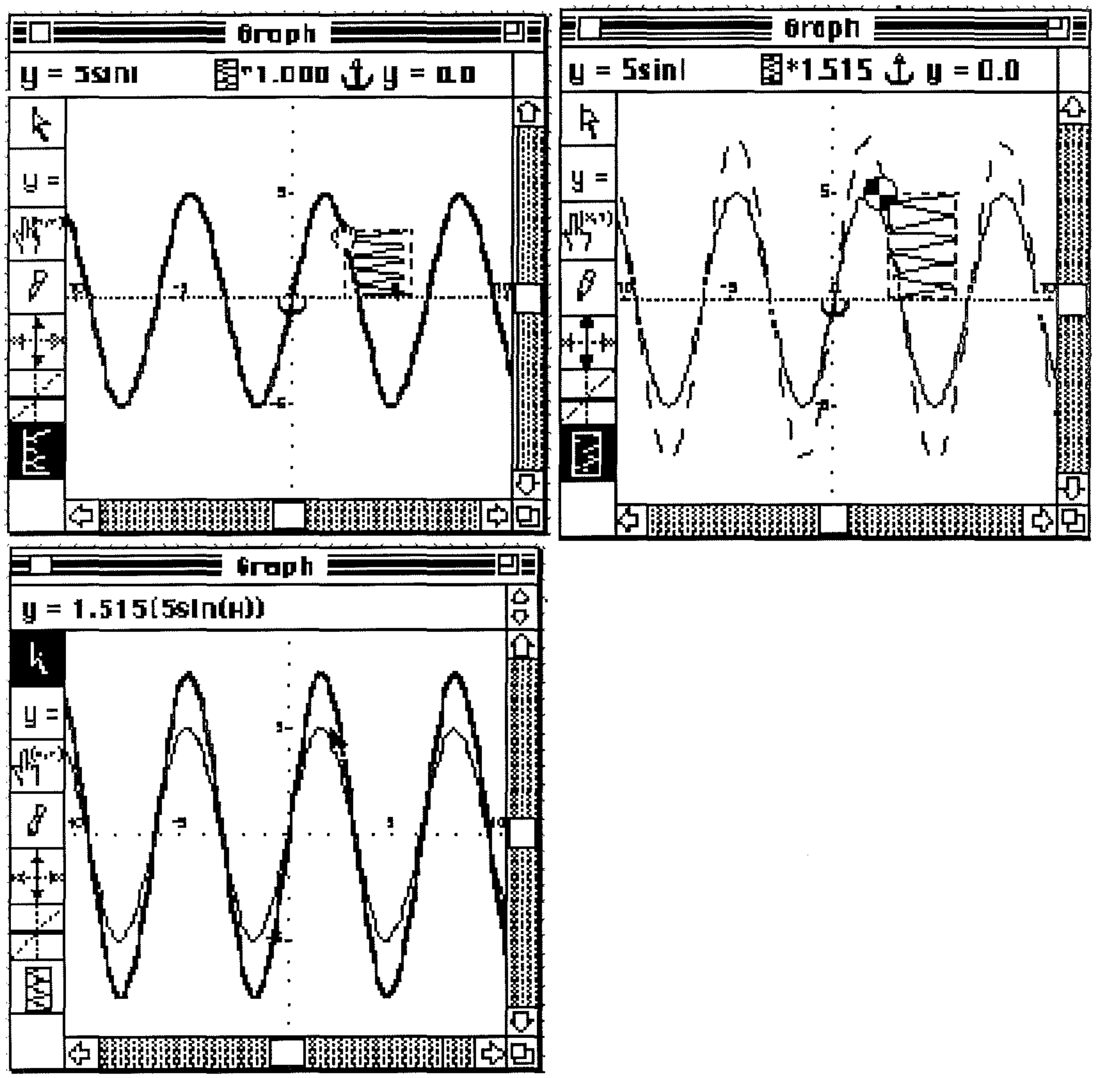


 - On clicking the mouse on *stretch icon* again student is asked to choose between *horizontal* and *vertical stretches*. These transformations allow students to stretch a graph in the direction of the x-axis or y-axis from a chosen line (anchor line). By choosing the anchor line in one of the axes, the effect of the transformation in an equation involving x and y is to have x or y multiplied by a constant, called stretching coefficient.

Vertical stretch with anchor line on the x-axis, for example, promotes a dynamic stretch of the value of y through the graph. In *equation*, the variable y is multiplied by the stretching coefficient which appears in the space of *feedback of commands*. Figure 1.4 presents *vertical stretch* of the graph of y=5sin(x) using the x-axis as anchor line by 1.515 in three phases.

Figure 1.4

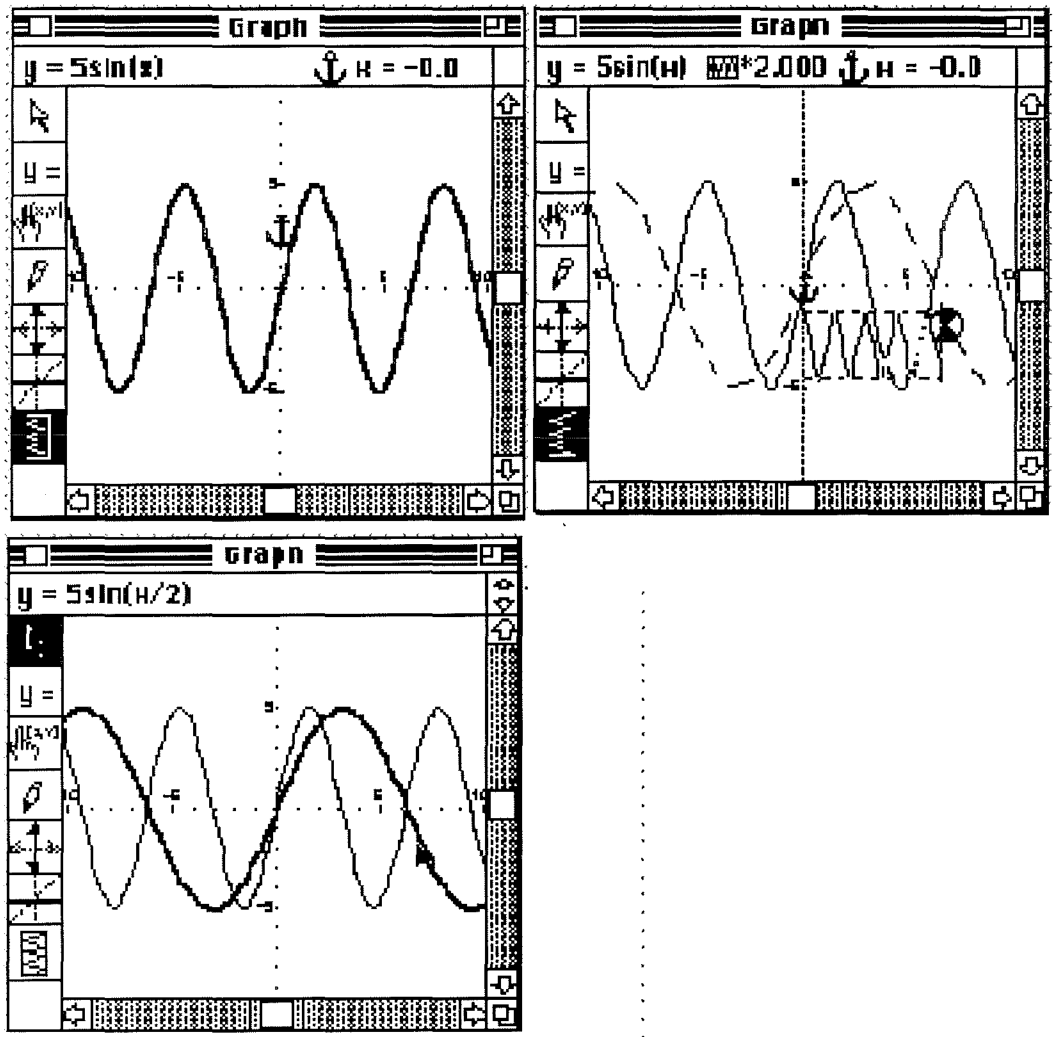
Vertical stretch of the graph of y=5sin(x), in three phases: after marking the anchor line; during stretch; and the result

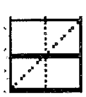


Horizontal stretch is similar to *vertical stretch*. They differ by the variable which each one of them stretches. *Horizontal stretch* stretches the value of x. For example, as shown in Figure 1.5, a *horizontal stretch* of 2 with anchor line in the y-axis in the graph of y=5sin(x), can change its period, but it maintains the amplitude of the graph while *vertical stretch* has the opposite effect.

Figure 1.5

Three phases in *horizontal stretch* in the graph of y=5sin(x): after marking the anchor line; during a dynamic stretch; and the final screen

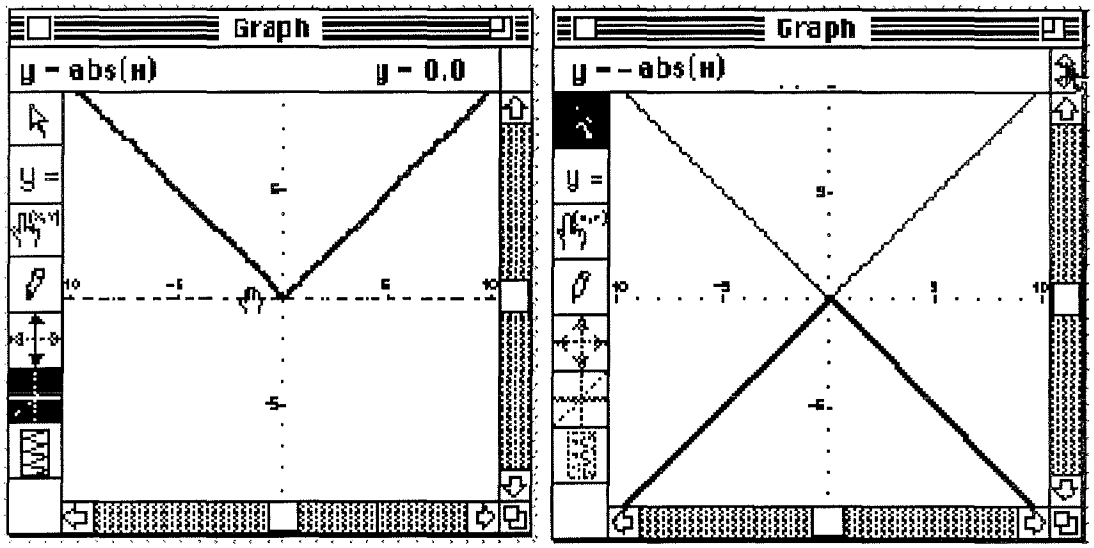



- On clicking the mouse on *reflection icon*, the student is asked to choose among: inversion, vertical and horizontal. Inversion (which will be not used in this research) reflects the graph through the line given by y=x causing an inversion of the function. For example, f(x) is reflected into f^{-1}(x).

Vertical reflection is a command to reflect the value of y with respect to a reflection line positioned horizontally. For example, by choosing the x-axis as reflection line, the value of y is reflected into -y. Figure 1.6 shows a *vertical reflection* of y=abs(x) with reflection line on the x-axis.

Figure 1.6

Vertical reflection of the graph of y=abs(x) in two phases: during the choice of reflection line; and its results



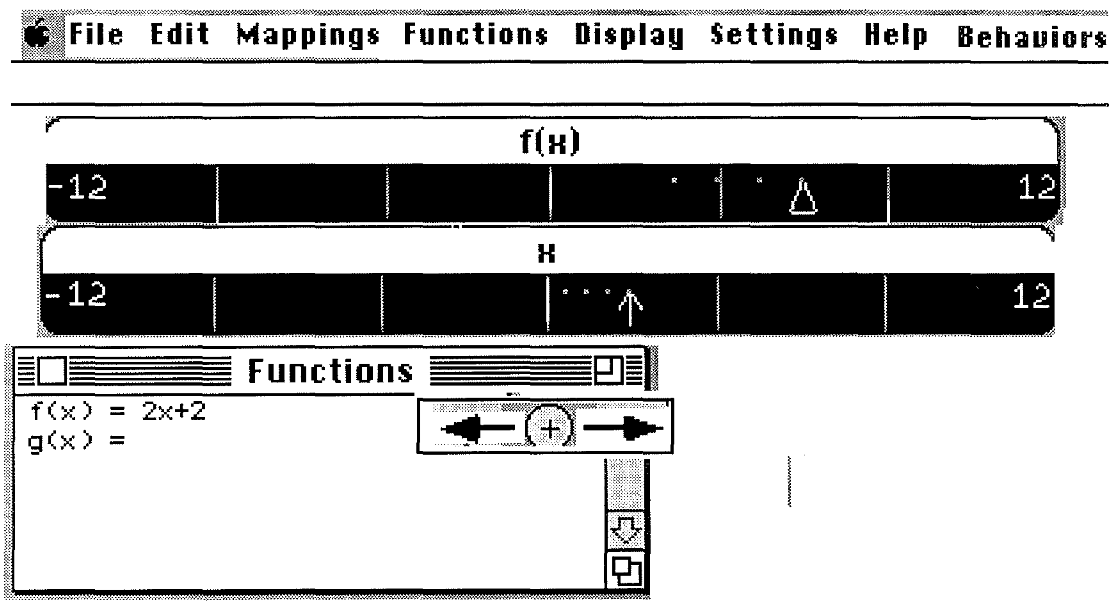
Horizontal reflection is similar to the vertical one. They differ by the variable reflected. For example, imagine a *horizontal reflection* in the graph of y=abs(x) with reflection line on the y-axis. It will not alter the graph.

2 Description of DynaGraph

DynaGraph (Goldenberg et al, 1992) is an educational software which presents a visual representation of function exploring the potential of dynamic manipulations of objects. It represents a function point-by-point by two sprites. One of them corresponds to the input of the function (in general denoted by x) and the second sprite represents the image of the function (f(x) or y). Using the mouse, the student moves (varies) x horizontally. Then, DynaGraph moves y according to the new position of x and the chosen function. It can explore one variable real function in three versions according to the position of the y-axis: (a) the axes are posed in parallel which I term parallel version, (b) the axes are posed in perpendicular disposition which I term perpendicular version; and (c) the axes are posed in Cartesian disposition, which includes a third sprite to represent the position of (x,y), which I term Cartesian version. Figure 2.1 shows the screen of DynaGraph with the parallel version on:

Figure 2.1

Screen of DynaGraph showing: R -> R and equation



The first stage in exploring DynaGraph is the definition of the function. Users are expected to enter the function by its algebraic representation. This step is done by selecting functions at the menu and writing down the equation. Then, DynaGraph enables the students to move the arrow, which represents x, and gives as feedback the change in the position of the triangle, which represents y. Thus, DynaGraph leads students to see function as the relation between 'transformation between x_1 and x_2'and 'transformation between y_1 and y_2' — a variational view.

Among the features of DynaGraph, I will emphasise here the following:

- 'the scales of x and f(x)' and 'the step x will vary' can be defined by the user. In figure 4.7, 'the step x will vary' is set to 0.5 units;
- the sprites of x, y and (x,y) have two modes: they can leave dots in the screen or not. In figure 4.7, DynaGraph is set to leave the dots;
- up to two functions can be explored in the same screen. When set to use two functions, DynaGraph presents another line (in the case of parallel version) to place the second function which is denoted in the screen by *g(x)*;
- the window called Functions can be set to be on or off, making the equation available to the users or not;
- functions can be explored by the user without knowing its equation. The behaviour menu allows a tutor to hide eight functions. The user can access these

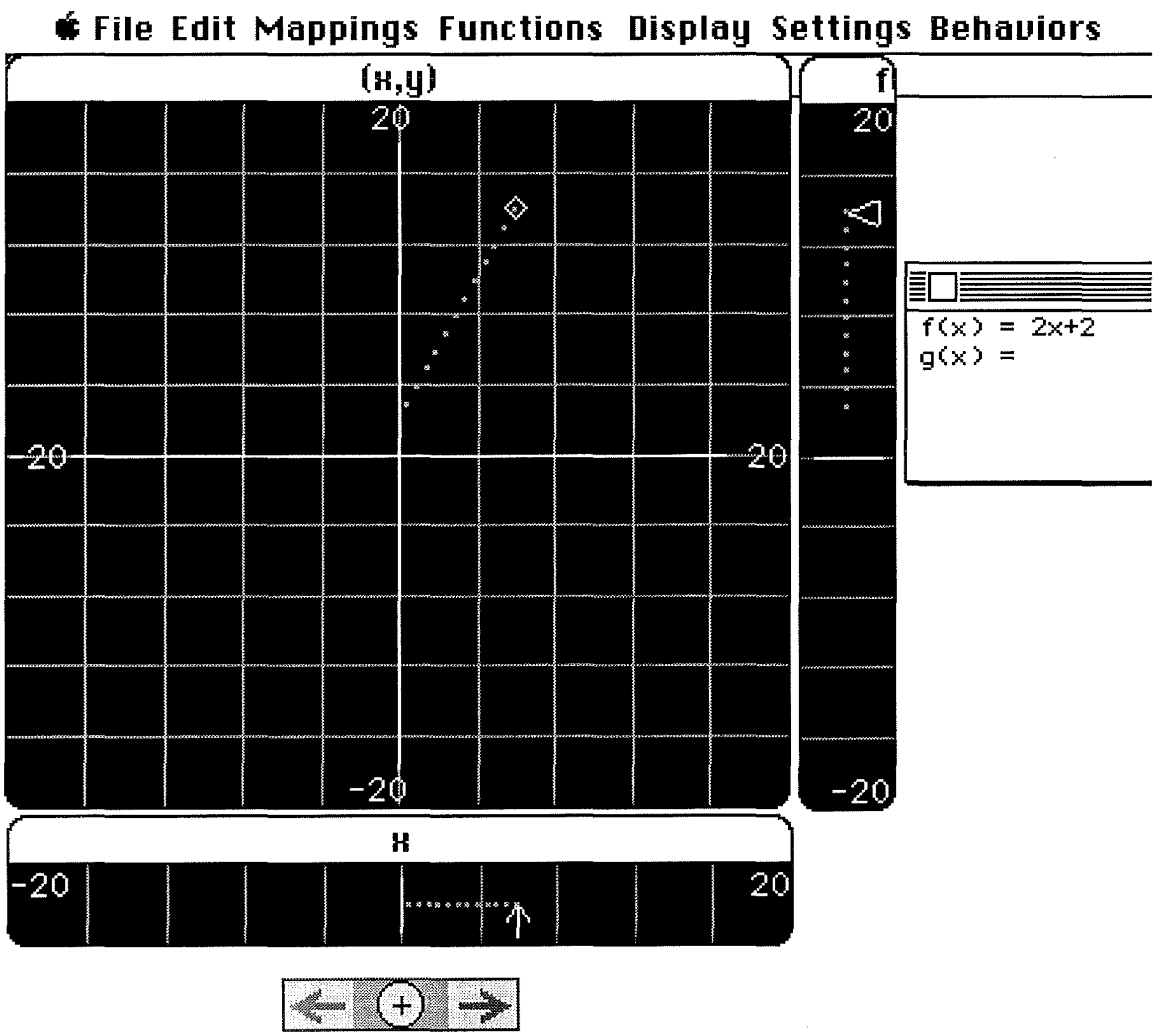
functions by selecting one of the numbered behaviours. In this case, the Functions window will not present the equation of the current function. There, the equation stays as the last defined equation;

- all the features are available to users' choice by the menu.

Figure 2.2 shows the Cartesian version of DynaGraph with the features in the following states: 'the step x will vary' is 0.5 units; the Functions window is on; the sprites are leaving the dots.

Figure 2.2

Screen of DynaGraph with the Cartesian version displayed



III — Review of the Literature on Function

This chapter starts by introducing in sections 1 and 2 terms and approaches I will use for representations and concepts. Then, section 3 reviews the literature on students' perceptions of a concept particularly the perceptions of the function properties. Sections 4 and 5 discusses the use of software in the topic of function particularly the use of Function Probe and DynaGraph and the final section presents the research questions in the context of the software programs.

1 Representations

This section will introduce the approach taken in this research to the meaning of representation and its relation with concept. The different representations of function used in school mathematics will also be discussed.

1.1 Representations and Concepts

Representation has been considered in mathematics education as a key to the construction of mathematical knowledge. I will base the definition of my use of the term on the survey published by Goldin (1992) in the proceedings of PME, which summarises the use of the term 'representation' in mathematics education research. Goldin classifies the meanings used for representation in mathematics education into three different types. The first one is internal, the second and third are external.

Internal representations "include individual representations of mathematical ideas as well as broader theories of cognitive representation..." (Goldin, 1992: 11). Goldin (1992) divides the external representations into two types: symbolic systems and contextual representations. Symbolic systems "can include linguistic systems, formal mathematical notations and constructs, or symbolic aspects of computer environment" (p.11). Cartesian Graphs, formulae, tables, and diagram are examples of formal mathematical notations of function. Regarding contextual representations, Goldin (1992) expresses them as being "external, structured physical situations or sets of situations, that can be described mathematically or seen as embodying mathematical ideas" (p.11). As the present study concentrates on symbolic systems, that is the meaning which will be used here for the term representation.

A first point to consider is the importance in mathematics of representation. Mathematics has a dual nature: it is a body of knowledge and a language. Therefore, as a language it has to be represented to communicate both to yourself and to others and to "provide an organizational framework" (Kaput, 1992: 522). Thus, representations are tools to facilitate both the understanding and the retrieval of mathematical knowledge. They are also used as a tool to universalise mathematics. These two characteristics of mathematics are regarded by many as inseparable. Dufour-Janvier et al (1987) suggest that mathematical concepts and representations are so closely associated that: "it is hard to see how the concept can be conceived without" (p.110) the representations. Thus, "the idea of representation is continuous with mathematics itself" (Kaput, 1987: 25).

Looking at the dual nature of mathematics, Kaput (1992) defined two worlds: "(i) a world of mental operations which is always hypothetical, and (ii) a world of physical operations, which is observable" (p.522). These two worlds can interact in both directions. Representations are part of the world of physical operations while concepts belong to that of mental operations. He defines a representation in two ways: in a functional way and in a technical way. In a functional way, the representations can be seen as a "system of rules (i) for identifying or creating characters, (ii) for operating on them, and (iii) for determining relations among them" (p.523). In a technical sense, a representation is "a set of rules that define the objects of the notation system and allowable actions on them" (p.523). In order to define the actions allowed in one representation, the material world where it is defined is essential. The material world can be paper-and-pencil, computer displays, physical objects, and so on. The actions are: transformations of objects within one representation and translations between objects from different representations. Translation between different representations is directional. For example, one can translate an equation into a graph by plotting points or translate information from a graph to find out an equation.

In this approach Kaput (1992) separates concept from representation, an approach also adopted by many mathematics educators (Greeno, 1983, Kaput, 1986, 1991, 1992, Schwarz & Bruckheimer, 1988, Janvier, 1987a, 1987b). Despite having the same starting position, while working with the concept of function, Schwarz & Bruckheimer (1988) argue that "Although the concept of function and its subconcepts are not theoretically linked to a particular representation ... the properties of a function are often understood in their representational context only and no abstraction of these properties is made by the beginning students" (p.552). This argument, in my view, shows the unfeasibility of disconnecting concepts and representations. I will therefore adopt an alternative notion of concept offered by Confrey et al (1991b) who take the position that "representations and ideas are inseparably intertwined. Ideas are always represented, and it is through the interweaving of our actions and representations that we construct mathematical meaning" (p.17). Thus, this research takes as a starting point the assumption that the connections between perceptions of a concept in different representations are essential for the construction of this concept.

Even if concepts and representations are inseparable, the successful use of any representation is not straightforward. Mathematics educators (Dufour-Janvier et al, 1987; Boulton-Lewis & Halford, 1990; Greeno, 1983; and Goldenberg, 1988) have focused on students' difficulties when using representations, and have argued that each representation has its own structure and ambiguities (Goldenberg, 1988). This means that students' perceptions of a concept must be investigated with due consideration of the nature of the representation. Boulton-Lewis & Halford (1990), for example, considered that "The choice, and successful or unsuccessful use, of a representation depended on the child's knowledge of the representation itself, of content and of appropriate procedures" (p.203). In my view their consideration draws attention to the fact that while examining the students' perceptions of a concept, one has to consider any difficulties inherent in the representation. Goldenberg (1988), for example, focused on students' difficulties while analysing Cartesian representation. Subsection 1.2 will present a review of students' difficulties in the use of representations of function particularly the algebraic and the Cartesian representations.

The use of more than one representation for each concept has been discussed by mathematics educators (Goldenberg, 1988; Confrey, 1992a). Goldenberg (1988) presents as view common among mathematics educators that "each well-chosen representation conveys part of the meaning best; together, they should improve the fidelity of the whole message" (p.136). For example, to perceive the symmetry of real functions is easier in the Cartesian representation than in the algebraic one (Confrey, 1992a). Nevertheless, if the study of symmetry is only derived from exploration in the Cartesian representation, the students can be led to limit their perception to a pictorial view without analysing the relation between x and y. To complement the previous argument defending the use of more than one representation, two other ones will be summarised here. The first is that same concepts can be presented in some representations but not in others. For instance, Euler's function that associates each rational number to 0 and the other numbers to 1 can be represented in the algebraic system, but it cannot be represented in the

Cartesian system. A second point to be considered is that each concept is perceived in different ways in each representation. Thus, developing a concept in different representations means that different aspects of the same concept can be perceived and leading students to generalise the concept to a wider range of applicability may result in overcoming limitations in each individual perception.

Students have to cope not only with different representations but also with making connections between different representations. Researchers (Confrey, 1992a; Borba, 1994; Goldenberg, 1988; Artigue & Dagher, 1993) discuss the use multiple representations. A common viewpoint is that inside each conventional representation a concept is seen in a different way. Artigue & Dagher (1993) argue that "A mathematical concept is not a monolithic object. A single concept may be understood from several points of view and may have several different representations; in mathematics one needs to be able to move freely between these points of view and representations, adapting them to the setting in which a concept is used" (p.1).

Confrey (1992a) summarising research on multiple representations makes several points in its defence. I intend in this study to investigate some of these points. She argues that multiple representations have the potential to:

- "highlight different aspects of the concept";
- "Lead to a convergence across representations that may improve or strengthen our depth of understanding";
- "promote examination of the potential conflict among forms of representations";
- allow assessing how changes in one representation affect another;
- "illustrate how alternate forms of actions in a representation can cause students to develop diverse schemes";
- "provide situations for students to conduct their own investigations of ideas";
- "provide opportunities for feedback, revision, and reflection that are created by the student" (p.149-150).

In an approach which considers that conceptual understanding arises from making connections across different representations, the main interest is to investigate whether the use of multiple representations leads to some convergence across representations. Two different possibilities can be seen; either two different forms of a concept derived from different representations can be connected by the students, or these forms remain isolated from each other. Then, some questions arise: In which conditions are connections spontaneously built? Which sort of activity must be undertaken to lead the students towards making connections? Can bridges be built to promote connections? Should the tutor build bridges to encourage students to make connections? Kaput (1992) suggests two activities to motivate students to make connections: match corresponding objects in different representations, and predict the effects of a transformation of one object in one representation to its corresponding object in another representation.

Moschkovich (1993) showed that the development of students' perceptions is a process which involves limitations which are not confined to students' perceptions. Teachers also can carry limitations in their perceptions. The case study reported by Speiser & Walter (1994) is a good example of how a teacher can identify limitations in the bridges they build for the students by listening to them. Starting with a contextual representation of a function as the frames of a cat walking, the teacher tried to make the students reach the concept of derivative as the limit of secants in a Cartesian representation. In a first step, the students constructed the concept of rate of change. Then, they pointed out that the bridge proposed by the teacher to connect rate of change in this example with the limit of secants did not make sense. The students argued that the initial representation presented only discrete points which could not be modelled by a function without considering a margin of error. With this example, I argue that allowing the students to freely navigate on different representations, can:

- help them to recognise any limitations in their perceptions of a concept in one representation,
- allow them to construct perceptions within a representation,
- encourage them to generalise these perceptions, and
- lead them to overcome any limitations of their previous perceptions.

While analysing how a concept appears in different representations, Moschkovich (1992) introduced the idea of looking at the status of the properties in each representation. She examined students' perceptions of the concept in one representation classified according their special status. To clarify this idea, I will refer to her example. She argued that in the same way that a root — the point at which a graph intercepts the x-axis (x-intercept) — has a 'special status' (a special point) in the Cartesian representation, the slope has a similar status in the algebraic representation for linear functions — linear coefficient. Therefore, the properties which can be recognised by coefficients assume a special status in the algebraic representation. Also, one property can have special status in one representation but not in another. This constitutes the asymmetry between representations. For example, slope has no special status in the Cartesian representation and demands from the students a variational interpretation of graphs which is not straightforward. In the same way, roots do not have a special status in an algebraic representation; one must make calculations in order to find out the roots using equations.

Asymmetry amongst the status of the properties constitutes a qualitative difference between representations. The use of qualitatively different representations is put forward by Lesh et al (1987) and Arcavi & Nachmias (1989) as a way to help students improve their perceptions of mathematical concepts. Arcavi & Nachmias (1989) analysed pupils and adults who were considered to be mathematically expert exploring a non-conventional representation. They observed that these individuals started to re-examine their previous perceptions in graphical and algebraic representations. The researchers raised the following question: "The role of a representation of a mathematical idea seems to go beyond the mere goal of having a tool to handle that idea. Could it not be that by introducing a new representation, we are not only establishing a way to express an idea or a concept, but also re-examining and consequently learning "more" about those ideas and concepts" (p.84)? This research involves the use of qualitatively different representations, that is different representations which attribute different status to the same concept. The requirement of incorporating qualitatively different representation can be justified by the argument that using different representations which give the concepts the same status will lose the opportunity of provoking students to re-examine their perceptions.

In order to promote the forging of connections, two points suggested by previous researchers will be investigated in the present study. Firstly, Moschkovich (1992) puts forward one important requirement for enabling students to connect information from different representations - students must recognise that the same property can have a different status in different representations. She found out that the students used properties perceived with same status in different representations as being correspondent. Secondly, by analysing the students' perceptions of concepts (such as derivative, continuity, limits, integration) in a clinical interview, Ferrini-Mundy & Graham (1994) suggest that the ability to co-ordinate algebraic and graphical representations may differ substantially across concepts.

On analysing the ways students made connections between perceptions of a concept in different representations, Schwarz & Dreyfus (1993) introduced two kinds of connections: simple connections and integration of information. By simple connection they meant direct links between two objects in different representations; for example, a student can link the direction of a straight line to the sign of the coefficient in linear functions. While integrating information, the knowledge built inside one representation serves to improve the knowledge of another. Thus, one question arises: does the use of qualitatively different representations lead students towards the integration of information? If so how?

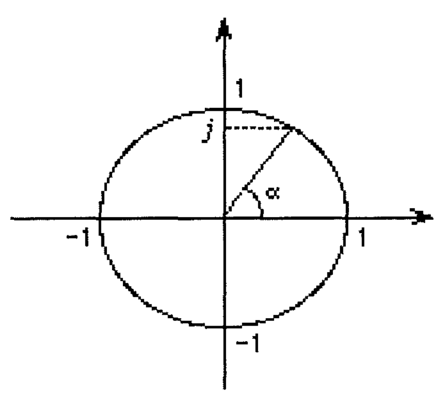
1.2 Representations of functions

When referring to 'school mathematics' in this work I mean 'traditional school mathematics' as taught in the majority of Brazilian secondary schools. A similar approach is taken by most North American high schools and Israeli secondary schools. In this section, I will survey the different potentialities and limitations of each representation of function used. The analysis of problems and advantages of each representation will have two foci: the first is the analysis of specific properties; and the second is the way students analyse the properties in each representation: pointwise, pictorial, variational or global. A discussion of the first type will be postponed to section 3 while the second focus will consider the ways students analyse functions represented by graphs and by equations.

The concept of function has been expressed in several different representations, for example, as equations and graphs. School mathematics has maintained the same multiple representational approach to exploring function, using representations such as: equations, graphs, diagrams and tables. In order to start discussing the advantages and problems in using these representations, four examples of real functions will be shown using these four representations: (*f*) the function which associates a number with its opposite; (*g*) the function which associates a number with its square; (*h*) the function which associates a number with the fixed value 2; and (*j*) the function which associates the value of an angle with its projection on the y-axis in the trigonometric circle as shown in figure 1.1.

Figure 1.1

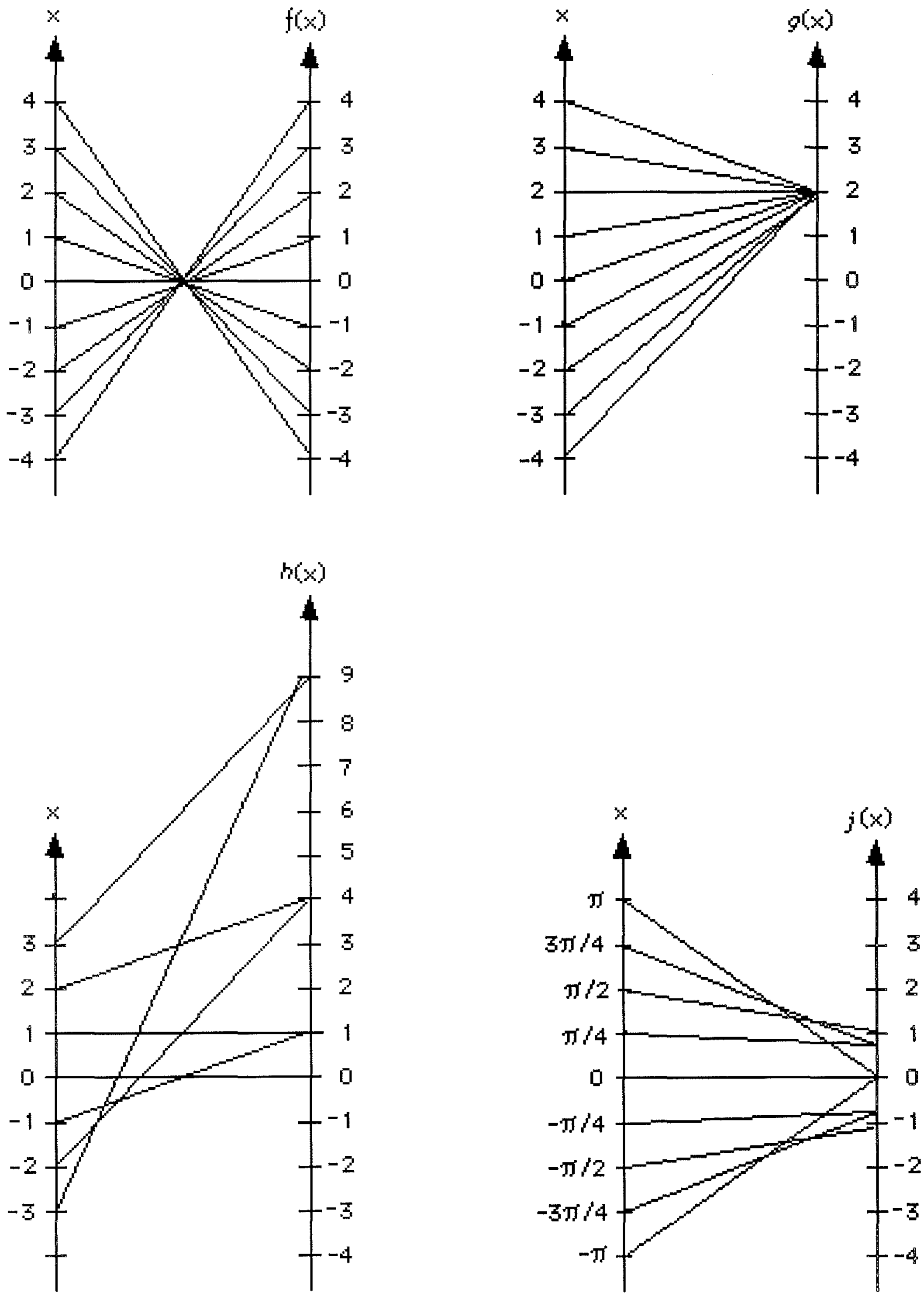
Projection of an angle α on the y-axis by the trigonometric circle



A diagrammatic representation frequently used is illustrated in figure 1.2.

Figure 1.2

f, *g*, *h* and *j* represented by diagrams



In tables the functions can be represented in the following way:

Figure 1.3

Tabular representation of the functions *f*, *g*, *h* and *j*.

x	*f*(x)
-4	4
-3	3
-2	2
-1	1
0	0
1	-1
2	-2
3	-3
4	-4

x	*g*(x)
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16

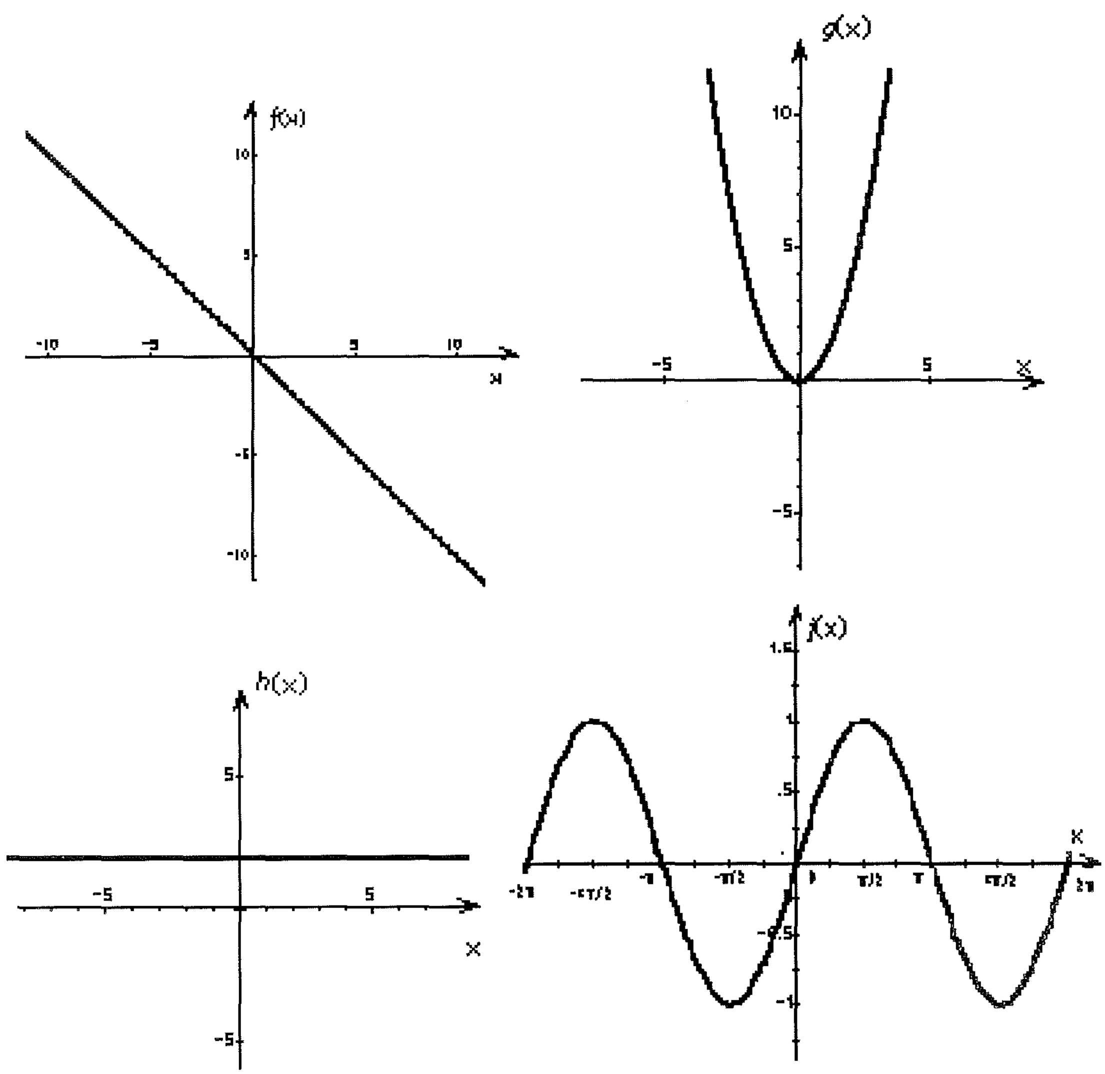
x	*h*(x)
-4	2
-3	2
-2	2
-1	2
0	2
1	2
2	2
3	2
4	2

x	*j*(x)
$-\pi$	0
$-3\pi/4$	$-\sqrt{2}/2$
$-\pi/2$	-1
$-\pi/4$	$-\sqrt{2}/2$
0	0
$\pi/4$	$\sqrt{2}/2$
$\pi/2$	1
$3\pi/4$	$\sqrt{2}/2$
π	0

The Cartesian representations of the four functions are shown:

Figure 1.4

Cartesian representation of the functions *f*, *g*, *h* and *j*



Finally, in the algebraic representation the functions take the following forms:

• $f(x)=-x$ for each $x \in \mathbf{R}$

• $g(x)=x^2$ for each $x \in \mathbf{R}$

• $h(x)= 2$ for each $x \in \mathbf{R}$

• $j(x)=\sin(x)$ for $x \in [-\pi,\pi]$; its domain can be extended to **R** by looking at any number z as being $z=x+2K\pi$ where $x \in [-\pi,\pi]$, and applying the function to x.

A comparison between the above examples shows that the algebraic representation brings precision. Students can know exactly the output value corresponding to any input. This precision is not easily obtained from graphs and diagrams, which only allow approximations. As for tables, the precision is obtained only for the inputs that appear in them, otherwise students will have to use interpolation. The Cartesian and the algebraic representations maintain the continuous character of the domain, while tables and diagrams can only represent discrete points. Summarising of the differences, Goldenberg (1987) reports that it is widely accepted that "Algebraic expressions specify the exact relationship, but give neither single examples nor a visual gestalt. Graphs provide a gestalt within the limits of the graph but leave precise details unclear. Tables provide examples of the mapping but do not specify its nature. ... taken together, multiple representations should improve the fidelity of the whole message" (p.197). The claim is that the more representations a student has, the better s/he can perceive a concept. I will investigate the truth of this as the key to the advantages offered by multiple representations lies in connections between them and also in different perspectives each of them can provide.

In order to examine how useful each representation is, this research investigates how problems and advantages of one representation might be both dependent on the particular content analysed and related to the ways students analyse functions. The first dependence will be addressed in section 3 while reviewing the research on students' perceptions of the function properties. Nevertheless, an illustration can be provided by the argument of Goldenberg (1991) about the use of graphs and equations while analysing linear functions. He shows that it is harder to understand linear function in the graphic representation than in the algebraic one. Goldenberg (op.cit.) suggests that "when approaching functions through their graphs, it may make most sense to begin with graphs that have no convenient algebraic representation and with notions that we typically ignore until the calculus, including the nature of the domain, local maximum or minimum, rate of change, and continuous or abrupt change" (p.81).

The second point investigated will be the core of the remainder of this section, where I will examine the different ways students analyse functions in different representations. The algebraic representation is in general taken in school to be the essence of a function (Confrey & Smith, 1992). This representation is explored by its potential to calculate the exact value for any element of the domain. Therefore, the procedural view of function tends to be the starting point. This approach can lead students to adopt a pointwise view when analysing a function through its equation and a variational one can be adopted following an analysis of the role of the coefficients of each equation (Janvier, 1983). For example, the linear coefficient of the equation 'y=ax+b' plays the role of the derivative and its sign indicates whether a function is increasing or decreasing.

If on the one hand, the equations lead students to a pointwise view of function, on the other, visual representations particularly the Cartesian one are claimed by experts in mathematics and in science to facilitate the interpretation of information, even of information related to variation. In contrast to this claim, Clement (1985), Preece (1983), Monk (1992) and Goldenberg (1988) show that the use of Cartesian representation has the potential to obscure as well as to clarify the concepts. The research on graphical understanding has pointed out that students usually interpret function properties from a graph by reference to its shape as a static picture (Goldenberg, 1988) - pictorially. Another way students interpret a graph is pointwisely. They come to see a graph as a tool to localise points (Monk, 1992). Considering both viewpoints, two aspects will be discussed:

- the ways the students analyse graphs;
- the possibility of analysing graphs in a different way.

In both analyses pointwise, variational and pictorial views will all be discussed.

Clement (1985) points out that one of the problems students have in interpreting graphs is that they see a graph as a picture. In this case, the shape becomes one of the features with special status in a graph. In a review of the literature on functions and graphs, Leinhardt et al (1990) report two ways students interpret graphs: considering the lines as a legitimate part of a graph, or considering only discrete points in a graph. Nonetheless, they point out that for both ways in general "the students often maintain a strict focus on individual points whether or not they are connected with a line. In other words, although lines are accepted as a legitimate part of graphs, they seem to serve a connecting function rather than possessing a meaning in their own right" (p.34).

Goldenberg (1991) goes further, pointing out that students usually observe only special points when interpreting graphs. Following his earlier study (Goldenberg, 1988) where bright students explore a graphic software while comparing two parabolas, Goldenberg (1991) concluded that the students used 'special points' or regions to interpret graphs, such as turning point and y-intercept. On looking only at special points and comparing different linear functions with the same coefficient 'b' at the equation 'y=ax+b', students can be led to connect the coefficient 'a' with the y-intercept as pointed out by Moschkovich (1992). Goldenberg (1991) points out that the gestalt way of interpreting graphs is a consequence of the way students learn about graphs.

Working with students' interpretation of graphs, Preece (1983) analysed students' perceptions of functions which require more than a simple reading of discrete points such as extreme values and derivative. She showed that 14-15 years-old students "have poor graph interpretation skills because they either do not understand the relevant concepts or have inadequate graph reading skills" (p.44-45). One of the errors she detected concerned the difficulties which students have in analysing function properties pointwisely. Preece (1983) detected that some students "were not able to answer questions about concepts which arose from the variables but which were not actually mentioned in the display, e.g. speed in distance-time graphs" (p.45). In my view, this can be interpreted as: the students who only interpret graphs pointwisely were not able to perceive function properties which do not have a 'special status' in the Cartesian graphs.

Apart from pointwise and pictorial ways of analysing graphs, a variational view can also be adopted. Tierney et al (1992) argue that to analyse function properties such as derivative and extreme values, students need to adopt a variational view. Nonetheless, they appreciate the importance of a pointwise analysis of other properties such as range and domain. The difficulty of developing a variational view is also another concern of Goldenberg (1993) who argues that when mathematical experts analyse a "Cartesian Graph and declare a function to be increasing over some portion of the domain, ..." they are "... seeing movement in a static picture, and using considerable interpretive skills that novices do not seem to bring" (p.13). Thus, the skill of reading a graph in a variational way is used by experts who claim that the use of Cartesian representation facilitates the interpretation of information. Analysing graphs in both variational and pointwise ways can facilitate the perceptions of different properties of function. Therefore, one aim should be to try to lead the student to a smooth way of developing a variational analysis of graphs (Tierney et al, 1992).

2 Methodological approach to investigating students' perceptions

This research will use the ways students describe functions as evidence of their understanding of the functions properties. 'Understanding' is at the cognitive level which is not observable, thus, this evidence cannot establish whether or not a concept is understood. To make clear that I am dealing with the observable world, these ways of describing the function properties will be called perceptions.

2.1 Two contrasting methodological approaches

The research on students' perceptions of function (Clement, 1985; Preece, 1983; Goldenberg, 1988; Mevarech & Kramarsky, 1993) has tended to concentrate on 'identifying' students' difficulties in developing the concept. The researchers try to detect common difficulties calling them misconceptions. Nesher (1987) defined misconception as "a line of thinking that causes a series of errors all resulting from an incorrect underlying premise rather than sporadic unconnected and nonsystematic errors" (p.35). Although in its origins the term has been considered to refer to "intelligent constructions based on what is more often incomplete than incorrect knowledge" (Resnick et al, 1989: 26), researchers into misconception have tended to concentrate on the negative aspects of the conceptions. Moschkovich (1992) discusses the analyses "of students' conceptions describing errors and misconceptions have focused largely on the "mis-" aspect of student ideas and have not considered conceptions that may be useful, applicable in some context, or productive for advancement" (p.129). Agreeing with her viewpoint, I argue that conceptions should most probably be analysed from a consideration of their potential for improvement, their origins, limitations and usefulness.

Moschkovich (1993) used the approach of 'alternative interpretations' which considers the positive and negative aspects of students' perceptions. According to her, the term "alternative interpretations" shows a certain respect for students' ideas by considering that there are alternative ways to conceive of a domain, although "there is a mathematically accepted way to think about the subject matter" (p.1). She points out that "misconception is no longer an adequate concept for referring to some of the conceptions that students generate" (p.1). In her research on students' use of x-intercept[1], she shows how the 'alternative interpretations' approach highlights the shortcomings of the 'misconceptions' approach.

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[1] x-intercept means the point where the graph intercepts the axis of x (x-axis).

As an example of this alternative perspective in her study of students' perceptions while connecting algebraic and Cartesian representations of slope, Moschkovich (1992) points out that the introduction of algebraic and Cartesian representations of linear functions with slope equal to 1 may obscure the difference between x-intercept and the independent coefficient[2]. She also claims that this introduction can be the origin of this connection. Nonetheless, Moschkovich (1992, 1993) showed that two of the students used x-intercept for slope, which was considered a "misconception", as a bridge to improve their perceptions of derivative. Moschkovich (1993) showed that the students refined the use of x-intercept in the following ways: (a) "The use of the x-intercept for b when" a=1 "was refined from using the x-coordinate of the x-intercept ... as the b in the equation, to using the opposite of the x-coordinate of the x-intercept ... as the b in the equation"; (b) "the context in which the use of the x-intercept is applicable was specified"; (c) "the x-intercept was explored as a reflection of the slope" (p.15). This shows that alternative interpretations can be useful in the process of developing a concept. It is important to understand this process in the construction of the concept as a transitional conception which has its usefulness, limitations, origin, and potential to bridge to more competent concepts. Therefore, in the 'alternative interpretation' approach listening to students is fundamental.

Following a similar approach, I will use 'associations' to describe what Moschkovich calls 'alternative interpretations'. The term association also includes students' perceptions of properties which merge with a different property. For instance, students can identify extreme values in parabolas and sines (graphs with turning points) but not in graphs of exponential functions (graphs without turning points). Thus, I will say that students' perceptions of extreme values are associated with turning points. Nonetheless, I do not mean that the students perceive extreme values and turning points as being the same property.

2.2 Obstacles

In developing an association students can follow two paths: either they recognise its limitations and improve their perceptions by revising it; or these associations become resistant to change and serve to limit the students' perceptions. Thus, associations can be transformed into knowledge-obstacles. As Artigue (1992) argues, "As far as some piece of knowledge has turned out to be successful in a wide range of situations, it becomes resistant to change, even if it must be at least strongly modified in order to cope effectively with new problems. This theory

[2] The independent coefficient in a linear equation 'y=ax+b' is given by 'b'.

implies that construction of knowledge cannot be totally continuous and error-free and that, behind resistant errors or difficulties, researchers have to look for the existence of some knowledge-obstacle" (p.110). Therefore, the present study must consider not only the path of students' developing perceptions of the properties of function but also the knowledge-obstacles present in this development.

Researchers (Sierpinska, 1992; Artigue, 1992; Dreyfus & Eisenberg, 1990) identify different origins for these obstacles. Sierpinska (1992), for example, searches in the history of the concept for obstacles similar to those she has observed with students. Artigue (1992) and Dreyfus & Eisenberg (1990) both seek the nature of the difficulties in the school curriculum. The investigation of students' perceptions of the properties of functions cannot be separated from their previous knowledge, particularly as it is derived from school mathematics. Therefore, while analysing the knowledge obstacles, the present research will analyse:

- similarities between associations developed by students in each microworld and the school approach;
- the obstacles derived from the ambiguities and structure of each microworld.

In section 3, I will discuss the patterns of students' difficulties which have been referred to in the literature.

2.3 A model for analysing students' perceptions

Researchers (Hoyles & Noss, 1987, 1993; and Sierpinska, 1992) have been working with a model to analyse students' understanding which classifies the acts of understanding into four categories: Using, Discriminating, Generalising and Synthesising. 'Using' is the act of using a concept as a tool for the functional purpose of achieving particular goals. 'Discriminating' is the act of explicating different parts of the structure of a concept. 'Generalising' is the act of extending the range of applicability of these parts. In the process of generalising, new aspects of the structure of a concept are discovered. Finally, 'Synthesising' is the act of integrating different representations of the same knowledge in different symbolic forms derived from different domains into a whole. Thus, conceptual understanding arises from making connections across different domains.

While Hoyles & Noss (1987) explore the model in which the first phase is 'using', the other authors begin with 'Identifying'. 'Identifying' and 'Discriminating' are different mainly because in the first the student differentiates one object among others while in 'Discriminating' the distinction is made between two objects. Thus, these two phases seem to be very close because their acts in fact distinguish one 'object' as being 'a characteristic' of the concept. This research will use only three of the categories. 'Using' will not appear in the analysis because of the nature of the activities designed for the study. The students will be asked to describe functions in different microworlds, thus, they will start at the stage of 'Discriminating' properties.

I will explain the reasons that led me to adopt the DGS model to analyse the students' perceptions of the properties of function. First, this research investigates these perceptions through different representations embodied in different microworlds. Therefore, the analysis needed a model which could categorise acts of perceiving within and between representations. Second, as the study examines different properties of functions, I could not take a linear model. DGS is not linear, it is a spiral model which considers that students can be working simultaneously in different categories depending on the property as well as the representation considered. Also, the categories are not necessarily followed in ascending order. Third, students will examine the properties of function in an exploratory computer environment. Finally, this research tries to trace the path of students' perceptions of each property.

Although DGS is a model for analysing students' understanding, I will use it to analyse students' perceptions in the observable domain considering that perceptions are in fact evidence of understanding. Thus, it is crucial that I define which kinds of perceptions I am using as evidence of the acts of understanding. With this purpose, I will detail the three categories by adopting the role of a student who is asked to describe functions while exploring them in diagrams and Cartesian graphs. I will use figures 1.2 and 1.4 of subsection 1.2. (Discriminating) students start to isolate one characteristic of a function (or set of functions) as being a differential function property. For example, suppose that I notice that in the diagram of f (see figure 1.2) any two consecutive lines[3] cross each other. Therefore, I discriminate 'two consecutive lines crossing each other' or not as being a property of this diagram. This is a perception which is particular to f when built within the diagrammatic representation. (Generalising) students start to recognise common patterns of a property they had already identified in some examples using one representation. Therefore, they adapt the perception to include the new samples. For example, 'two consecutive lines cross each other' can also be observed in diagrams of h and j (see figure 1.2) restraining the domain. For h, 'the consecutive lines cross each other' between -3 and 0, while 'the consecutive lines do not cross each other' between 0 and 3. Therefore, I generalise 'my' first perception of the property to characterise other examples of functions in diagrams. During this phase, I am still analysing acts within

[3] By consecutive lines I mean lines which start on consecutive numbers.

one representation. In this aspect the model I am using differs from the one the authors use. They consider that generalisations can also be made between different representations. I will analyse these, as well as modifications in previous knowledge, as being evidence that students are 'Synthesising'. For example, suppose that I have already discriminated and generalised the direction of the graphs for all the graphs of figure 1.4. While trying to compare diagrams and Cartesian graphs, I realise that 'two consecutive lines crossing each other' and 'the direction of the graph is north-west to south-east' comprise the only one function property.

While working with the UDGS model, Hoyles & Noss (1987) created a situated abstraction/scaffolding framework to analyse knowledge construction. They had observed that "students frequently construct and articulate mathematical relationships which are general within the microworld yet are interpretable and meaningful only by reference to the specific (computational) setting" (Hoyles & Noss, 1993: 84). For these relationships they coined the term 'situated abstractions'. Their concern is centred "on the ways in which learners structure their own learning, as well as on the ways in which the setting structures it" (Noss & Hoyles, 1996: 108). This led them to work with the scaffolding metaphor.

The scaffolding metaphor used by Wood et al (1979) was extended to computational settings. The original idea referred to the "graduated assistance provided by an adult which offers just enough support (and no more) when needed so that a child can voyage into his/her *zone of proximal development*" (reference to Vygotskian theory as cited in Hoyles & Noss, 1993: 85) . On extending the term, Hoyles & Noss (1993) focused on the setting, on the symbolic system used to represent the concept, "more particularly, the extent to which the scaffolding mechanism is domain contingent" (p.85). The extension also diverges from the original meaning because the assistance is controlled not by the judgement of the tutor but by students' interaction with computer environments. Thus, the medium led to students developing their own path of learning.

Although Hoyles & Noss (1993) built this framework while working with computer environments (in particular with microworlds), they argued it can be used in other contexts. Therefore, I discuss here the possibility of using this framework in the context of representations, in particular of formal mathematical systems. They point out, for example, that school algebra is not a constructive language, because algebra has been taught with a view to legitimate mathematics. They call for the construction of computer environments, which we can recognise as mathematical, "in order that students can exploit them as scaffolding for the articulation of situated abstractions" (p.90). I will use the framework while analysing students' perceptions in different representations embodied in microworlds. These microworlds were designed with activities around software programs.

The role of building these situated abstractions has been discussed by researchers (Hoyles & Noss, 1987, 1993; Gurtner, 1992). A common view is that situated abstractions lack universality. The students perceive the concepts inside one medium and in a different medium they will build other perceptions. Hoyles & Noss (1993) defend these processes with the argument that they can be "constructed by a learner who may have no access to the semantics and syntax of general mathematical language" (p.84). I conjecture that even to students who already have the semantics and syntax of mathematical language, the construction of situated abstractions isolated from their previous knowledge can lead them to perceive properties in a wide range of applicability of the concept. Therefore, it can be fundamental in overcoming limits of associations when synthesised.

In this process some natural questions still remain: Can these situated abstractions be synthesised with mathematical knowledge, or among different media? Should the tutor build the bridges for these syntheses? Under what conditions do spontaneous syntheses occur for students? Gurtner (1992), in his article using the bridge metaphor, argues that contextual environments need to be used in order to help students 'transfer' mathematics to these environments. Therefore, he expects the teacher to build this bridge. On the other hand, Moschkovich (1993) argues if teachers build the bridges for the students, it is more likely that limitations will be perpetuated.

3 Students' perceptions of the function properties

The following function properties were chosen as foci for the investigation of students' perceptions of functions: turning points, variation, range, symmetry and periodicity. This section will discuss the epistemology of each of these properties and the criteria of selection adopted and the knowledge-obstacles reported in the literature.

Three criteria were used to select the properties. First, as I consider that the understanding of function requires a diversity of forms of analysis, I decided to focus on the properties that could allow the study to cover this multiplicity: pointwise, variational, global and pictorial. Second, I investigated properties which the students had already met in school mathematics. Thus, the selected properties were emphasised in the families of functions already studied by the pupils: linear, constant, quadratic and trigonometric functions. The third criterion concerns the particularities of each microworld. The properties chosen were either the ones considered by the researchers to be easily perceived by the students when using FP and DynaGraph, or the ones I believed to be hard to perceive in these microworlds.

diSessa (1995) argues that epistemology is one of the pillars on which to design exploratory learning environments. He claims that in designing these environments one "must take the epistemology of instructed disciplines seriously, but part of our strength is in the innovative perspectives we can bring to bear on subject matter" (p.28). Following his argument, the present study takes into consideration the expectations of the designers of FP and DynaGraph. These expectations are included in the third criterion of the choice of the properties. At this point, it is not possible to discuss these expectations because such discussion will lack the review of the literature using these programs. Therefore, I will postpone it to section 5. On the other hand, to start the discussion of the epistemology of each of the properties, I will consider their epistemology in mathematics. The epistemology of these properties in the school approach must be considered because this work focuses on investigating the perceptions of these properties by pupils who had already studied them at school. This epistemology will be discussed in two ways: by reviewing the literature on students' perceptions of each of the properties and by analysing the epistemology adopted in the school attended by the pupils from the sample. The first one will be developed in subsections 3.1 to 3.5 and the second will be the object of one chapter of the analysis.

3.1 Turning points

Turning point can be defined as "A local minimum or maximum point on a curve, at which the ordinates cease increasing and begin decreasing or vice versa" (Glenn & Littler, 1984: 214). In the case of differentiable functions, turning point is the point where the derivative of the function is zero and the derivative changes sign. Observe that from a mathematical viewpoint the notion of 'local' is fundamental to the concept of turning point. Local cannot be perceived in a pointwise way. Students have to see the function in a whole interval or whole domain. This is usually called a global view. The notion of 'local' is not discussed or introduced in secondary education. Moreover, in the curriculum local is suppressed. Thus, turning points are seen as global maximum or minimum. With regard to the second part of the definition, turning point can be analysed in a variational way.

Among those researchers who discuss students' perceptions and the curricular approach of turning point, there is a consensus that this property is explored in school and thus perceived by students as a 'special point' in Cartesian system. Confrey (1992a), for example, points out that turning point is a 'special point' emphasised in the family of quadratic functions. On discussing the emphasis for the quadratic functions, she argues that in the curriculum functions are treated in families with emphasis on special points. For quadratic functions, for example, she mentions the roots and the turning point. Goldenberg (1988), examining bright students exploring a graphic software to match a parabola presented to them by a graph with equation, concluded that students use 'special points' or regions to interpret the graphs such as turning point and y-intercept.

3.2 Variation

Variation was divided into four properties, which I will call: constant function, monotonicity, derivative, and second derivative. Although variational properties can be seen as a whole, school mathematics treats them compartmentalised. In this research, I will analyse these properties separately.

3.2.1 Constant function

Constant function is defined as "a function f for which there is an object such that $f(x)=a$ for all the domain of f" (James & James, 1968: 73). It can also be seen as a function in which the output does not change.

Researchers in the topic show that constant functions can be seen in different ways depending on the representation. Each of these ways involves different problems. In Cartesian system (see figure 1.4) it can be seen as a horizontal straight line. Thus, constant functions can be pictorially characterised in graphs. In algebraic representation, it usually appears as the absence of x in an equation. This form is reported to be a problem in students' perceptions. For example, by analysing the results of a questionnaire with A-level students, Bakar & Tall (1991) concluded that the absence of x in an equation led the students to consider that it was not a function. Nonetheless, an alternative equation where x is present (y=0x+b) led the students to consider it as a function.

Connections between verbal description and Cartesian representation of constant function are usually reported to be a problem. Working with students without previous knowledge of functions, Mevarech & Kramarsky (1993) detected five lines of thinking in analysing graphs which have consequences on students' construction of graphs of constant function from a qualitative verbal description. One of them is significant for this study: some students think of graph as 'a single point', representing all situations as a point. Nonetheless, on realising that a change cannot be represented by a single point, they change their representation of increasing or decreasing linear graphs for two points, many times in different graphs. Thus, constant functions continued being represented by a single point in graphs. They give two different reasons for that: the emphasis on ordered pair in school and the intuitive sense that the final point is of most interest. The stage of constructing a constant function graph as a single point is a common behaviour in students' perceptions in graphs (Goldenberg, 1988). He concluded that students use graphs by the points and do not interpret the line between two 'special points' as being formed by points. Therefore, the end points are the important ones. Thus, the pointwise view is usually considered only for special points.

3.2.2 Monotonicity

The monotonicity of a function is usually classified as increasing and decreasing function. The idea of constant function can also be seen as the stage between increasing and decreasing. An increasing (decreasing) function is the one "whose value increases (decreases) as the independent variable increases" (James & James, 1959: 102 and 200). In Cartesian graphs the idea can be seen as "a function whose graph rises (falls) as the abscissa increases" (p.102 and 200). Therefore, the property of monotonicity requires a variational view of functions. Nonetheless, this property can also be pictorially identified by the direction of a graph. In the algebraic representation the idea of monotonicity can be seen by calculating different points of the function. In the case of linear equation, the idea also can be detected by the sign of the linear coefficient.

The idea of monotonicity was investigated by Hillel et al (1992) using the Computer Algebra Systems (CAS) in collegiate courses, particularly in courses on functions. They reported two kinds of problems in students' perceptions of monotonicity: the bi-directional sense of the line, which means that the students see the graph as starting at the origin and continuing in both orientations; the confusion of the referent interval, which means that the students were confused about whether they should use domain or range. Therefore, these findings suggest that the students have difficulties in comparing the behaviour of x and y and in isolating the variables in a graph. In other words, the difficulties are concerned with interpreting graphs in a variational way.

3.2.3 Derivative

One of the most frequently investigated function properties in mathematics education is derivative. Mathematically it is defined as being "the instantaneous rate of change of a function with respect to the variable" (James & James, 1959: 107), while rate of change is defined in the following way: "Let $y=f(x)$ be a given *function of one variable* and let Δx denote a number (positive or negative) to be added to the number x. Let Δy denote the corresponding increment of y:

$$\Delta y = f(x + \Delta x) - f(x).$$

Form the increment ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ " (p.107).}$$

Therefore, the derivative is defined by the limit of the increment ratio when $\Delta x \to 0$.

Secondary mathematics usually does not explore the idea of limit. Derivative is studied as 'slope' (for linear functions), rates of change such as speed, or as linear coefficient in a linear equation. A common claim is that to understand this property as slope and rate of change, a variational view of function is required from students. For example, investigating students' exploration of contextual problems in a multiple representational software, Confrey et al (1991b) argue that the students have improved their perceptions of derivative as rate of change by developing a variational way of analysing graphs and tables. After instructional sections on sequence of numbers, the majority of students built the notion of rate of change linked with slope of a graph. Moreover, some of them connected the straight disposition of the points in a graph with the constant rate of change.

On the other hand, research shows that students usually perceive derivative using pointwise or pictorial views. In his work on 'misconceptions in graphs', Clement (1985) points out three types of association presented by the students when interpreting graphs: height for slope; slope for height; and height for difference. Note that all these associations seem to be a consequence of the pointwise way of interpreting a graph. A different source of association pointed out in the literature (Goldenberg, 1988) is 'angle for slope'. The students interpret the slope of a linear graph as being the angle formed by a straight line and the x-axis. Thus, the students interpret the graph as a picture.

Despite working with the negative aspects of students' perceptions, their findings are important starting points for a qualitative analysis of students' perceptions of derivative. For example, in linear functions passing through (0,0) the students can see the slope for height. Nonetheless, this perception cannot be generalised to the other linear functions.

Note that all the above-mentioned investigations about derivative, apart from Confrey et al (1991b), deal with derivative for linear functions only. Another association pointed out by Clement (1985) is the slope as curvature. On trying to investigate the slope of curves, students rarely distinguish slope from curvature. While exploring the path of students' perceptions of non-linear functions, Speiser & Walter (1994) showed a gap between a pointwise way of perceiving rate of change and the perception which deals with the limit of secant lines. The limit version requires a global view of function. The students must analyse a function as defined in a non-discrete interval. These researchers analysed students' difficulties with derivative while working with the natural modelling of the motion of a cat given by discrete frames. They listened to 305 students while introducing the concept of derivative as tangent line. In the first class, they introduced "the derivative as a rate of change, beginning with a discussion of how we measure speed" (p.137). Secondly, they discussed instantaneous speed with the data from the motion of the cat. At the end, they asked how fast the cat was running in two different frames. The students showed the researchers that they could not work with a continuous transformation of secant line to a tangent line without considering a margin of error. Therefore, they demonstrated a gap between the pointwise and global view of derivative.

3.2.4 Second Derivative

Considering the derivative of $f(x)$ as the function $g(x)$, the second derivative can be defined as the derivative of $g(x)$. Therefore, it can be seen as the variation of rate of change. This property is usually studied in its graphical form as the curvature of a plane curved graph and in its algebraic form as the angular coefficient of quadratic equations. The curvature of a graph can be defined as "the rate of change of the inclination of the tangent with respect to change of arc length" (James & James, 1959: 95). Second derivative, like derivative, is a property which requires of the student a variational way of analysing a graph. This property was selected to be investigated because it is emphasised when the pupils study the family of quadratic functions in secondary mathematics.

Students interpret second derivative as the curvature of graphs using a pictorial view. Goldenberg (1988), for example, points out that the students were not able to compare curvature of parabolas without the same turning point. He argues that the graph leads students to the illusion that 'two parabolas distinguished by a vertical translation' have different curvatures. Another finding was reported by Clement (1985) when positing that students change slope for curvature. In other words, curvature and slope are usually mismatched by the students.

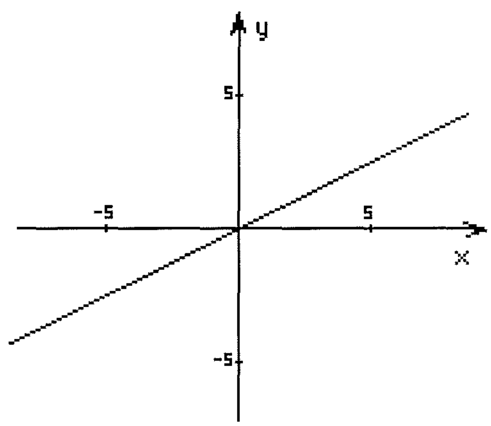
Nonetheless, a variational way of analysis can lead students to improve their perceptions of second derivative. Confrey (1992a) verified that the students developed a view of the dimensionality of a quadratic function where the difference of rate of change must be constant. While investigating students' use of tables in Function Probe exploring the quadratic functions in contextual problems, she pointed out the benefits of looking at the second derivative in a variational way.

3.3 Range

From a mathematical viewpoint the range of a function is defined as "the set of values the function ... may take on. The range of the function $f(x)=x^2$ is the set of all nonnegative real numbers, if the domain of the function is the set of all real numbers" (James & James, 1959: 323). Therefore, despite being linked with the image of x by f, range requires a global view of function. The students must see the function as defined in the whole domain.

Figure 3.1

Graph of $f(x)=0.5x$



One difficulty reported in the literature about the perception of range is the bounded representation of a boundless property. Goldenberg (1988) points out that students usually interpret graph considering only what is in the display. That is, the students do not extrapolate the screen of a graph. Thus, the range of a linear graph, for example, can be perceived as being limited (see, for example, figure 3.1).

3.4 Symmetry

The idea of symmetry is intrinsically a geometric idea (Confrey, 1992a). "A geometric configuration (curve, surface, etc.) is said to be symmetric ... with respect to .... a line, ..., when for every point on the configuration there is another point of the configuration such that the pair is symmetric with respect to ....[the] line " (James & James, 1959: 384). A pair of points is symmetric with respect to a line if, "the line ... is the perpendicular bisector of the line segment joining the two points" (p.385). Symmetry with respect to a line, I will express as 'line symmetry in...'. The pictorial view is in general used to detect line symmetry on graphs. Despite being a property usually explored by the shape of graphs, the idea of line symmetry can also be seen in the relation between x and y - in a pointwise way.

School mathematics usually explores symmetries with respect to a point or a line in the topic of function. For example, a parabola is line symmetric with respect to a vertical line called line of symmetry. The idea of line symmetry is usually studied while exploring the family of quadratic function as a qualitative property to characterise these functions. Confrey (1992a) called for a stronger emphasis on line symmetry, dimensionality and rate of change in the family of quadratic functions, instead of the emphasis on turning point and roots. Another example of symmetry is symmetric numbers. They are symmetric with respect to the point zero.

3.5 Periodicity

A periodic function of one real variable is defined as: "a function $f(x)$ such that the range of the independent variable can be divided into equal subintervals such that the graph of the function is the same in each subinterval" (James & James, 1959: 290). They also conclude that "the length of the smallest such equal subintervals is called the period of the function" (p.290). Note that as in school mathematics, this definition requires a pictorial perception of graphs. Nonetheless, a functional view can also be given by '$f(x)$ is a periodic function if, and only if, there is a real number 'a' such that $f(x+a)=f(x)$ for all x in the domain'.

3.6 Patterns in students' perceptions and school curriculum

As this study investigates students who have had some acquaintance with functions in school, school knowledge will clearly affect students' perceptions of the function properties. Moreover, the selected microworlds embody qualitatively different representations of functions and thus provide a good opportunity to compare obstacles students might face which arise from the school approach.

The school emphasis on algebraic representation is one aspect that can present students with obstacles. Artigue (1992), for example, concluded that "Beliefs and habits about the status and role of the graphic setting act as didactic obstacles and they have to be explicitly questioned in order to obtain the necessary epistemological changes both in teachers and students" (p.132). Although she investigated high algebra students, her findings can be considered at all levels of school mathematics.

In my view, the unbalanced emphasis on the use of representations by concentrating on the algebraic one can lead students to obstacles.

Dreyfus & Eisenberg (1990) called attention to compartmentalisation of knowledge as causing obstacles for students' perceptions of functions in visual representations. On analysing students' reluctance to visualise, they searched for both similarities and differences between school approach and thought process on algebraic and diagrammatic representations. They argued that this reluctance is independent of how the students are presented with the concept. In addition, they pointed out that "students seem to consider the visual aspects of a concept as something peripheral to the concept itself" (p. 27). They investigated curricular and cognitive viewpoints of this widespread reluctance. On the basis of the theory of "didactical transposition" (Chevallard, 1985 quoted in Dreyfus & Eisenberg, op.cit.), when "knowledge undergoes a fundamental change when it turns from academic knowledge as known by mathematicians into instructional knowledge as taught in school" (Dreyfus & Eisenberg, 1990: 29), they argued that in didactical transposition knowledge is compartmentalised in "bits of knowledge" to be put in a linear sequential way. This led students to have more facility to process the sequential information. On the other hand, "An analytical presentation, being sequential, is simpler to absorb — elements are presented one after the other, none are missed. Relationships between the elements may be lacking; if they are present they have to be introduced separately from the elements, tacked on to them. Diagrammatic representation is simultaneous, the elements and relationships between them are apparent at the same time, at the same location. They are therefore likely to be difficult to read, absorb, and interpret" (p.31). Artigue (1992) and Dreyfus & Eisenberg (1990) offer two different approaches which can be considered as the origin of the students' difficulties in working with visual representation.

Although the genetic epistemological analysis of obstacles is not a goal of my research, some obstacles to the understanding of function offered by Sierpinska (1992) must be taken into consideration.

- Regarding changes as phenomena, students focus on how things change, ignoring what changes. She exemplifies with the inability of referring to x and y in the Cartesian representation. The student does not see a graph as formed by points (x,y). In other words, the absence of a variational way of interpreting graphs seems to create barriers.
- Privileging the linear functions. This kind of obstacle is also analysed by Markovits et al (1983) considering the nature of the curriculum. Studying the

pupils' perception of function, from ninth grade in an Israeli school, in algebraic and graphic representation, they claimed that the mathematics curriculum should de-emphasise the linear functions and introduce a larger variety of function. In my opinion, the introduction of the properties through a variety of families of functions can allow students to recognise the invariants which characterise each property.

Schwarz & Hershkowitz (1996) also explored this emphasis in a comparative study between two groups of students following different curricular approaches to function. Despite working with curricula with a rich spectrum of functions, both groups differed by the activities and tools used: (a) one worked with computer environment and open-ended problems and (b) the other with ordinary activities and tools. They showed that the group (a) were able to use functions different from linear when necessary despite having a tendency to use the linear ones whenever possible. In contrast, group (b) used in almost all the cases the linear functions. Their results show the difference made to students' preference for linear functions when following different curricula, which suggests a curricular origin of the obstacles.

Schwarz & Hershkowitz (op.cit.) argued that "if prototypes are persistently too dominant, they impede learning, because they are used as frame of reference in the judgment of other examples" (p.259). Taking this argument into account, I will investigate its extension in relation to the function properties from linear preference to preference of the properties in other families of functions.

Finally, I would like to comment the patterns in students' preference for polarised knowledge reported by Artigue & Dagher (1993). They analysed 14-18 year-old students working in a multiple representational computer environment, focusing on their correlation of properties with special status in algebraic and Cartesian representation. Their findings showed that the students exhibited a persistent difficulty in ordering coefficients and an easy correlation of the signs of coefficients in the equations. The students preferred to explore knowledge when polarised such as positive versus negative and increasing versus decreasing.

In the present research, some questions regarding these obstacles will be addressed: Will these obstacles be observed in students' explorations of the microworlds? Will they be overcome, and if so, how? Will different obstacles appear? Can I trace similarities between the obstacles and the school approach to function?

4 Software for functions

The use of educational software in mathematics education must be investigated in two aspects: the technical potential of the software to help the development of concepts and representations reflecting the expectations of the designer and then the students' use of this technical potential. Thus, the environment created around this software is essential.

4.1 Technical potential of software for functions

Since the introduction of the computer in mathematics education, its dynamic possibilities have been increasingly used to explore the concept of function. Kaput (1992) points out that "Historically, mathematical notation systems have been instantiated in static, inert media, but the new electronic media now afford a whole new class of dynamic, interactive notations of virtually any kind" (p.522). The dynamic potential of the computer has been explored in many forms such as:

- conventional representations assume dynamic possibilities, as in Function Probe, in Algebra Toolkit (Schwartz et al, 1991), in Graphic Calculus (Tall et al, 1990) and in RandomGrapher (Goldenberg et al, 1992);
- the multiple representations of a concept gain dynamic interactive links, as in Grapher (Schoenfeld, 1990), in Triple Representation Model (TRM) (Schwarz & Dreyfus, 1993) and in Function Probe (Confrey et al, 1991a);
- new representations exploring the dynamic manipulations of objects can be created as in Function Machines (Feurzeig & Richards, 1991) and in DynaGraph (Goldenberg et al, 1992).

The dynamic possibilities of direct manipulations inside graphic representation of functions have been increasingly used in software. In early multiple representational software, graphs kept the status of display representations. The actions were in general produced in another representation and the software feedback was given in a graphic representation. Nowadays, software allows actions and feedback in different representations. For example, translations in functions are now permitted within Cartesian representation in software (in Function Probe and in Algebra Toolkit). With the dynamic manipulations now possible in earlier display representations, students can act within a representation by transforming objects (Kaput, 1992). Thus, the earlier display representations gain the status of action representation.

One of the actions possible in the new representational software is the dynamic transformation of graphs. In pioneer software, transforming graphs were made only by changing coefficient of equations. Nowadays, it is also possible to transform a graph directly using the mouse. When transforming graphs by changing the coefficient in the equations, students only had the starting and ending graphs. Goldenberg (1988) argued that in this way students had difficulties in perceiving the real transformation. For example, he pointed out that students usually perceive two parabolas translated vertically as having different curvatures. Therefore, students can conclude that the transformation of the equation $y=x^2$ into the one $y=x^2-5$ will change the curvature of the graph. Goldenberg (1991) claimed that students need to transform graphs within the graphical representations. He hypothesised that if the match of two parabolas by a vertical translation could be made directly from the graphs, students could change their way of measuring the congruence of a parabola from measuring their distance horizontally to measuring it vertically. Going a bit further, I argue that other ways of verifying curvature of parabolas can be created from these dynamic transformations. Thus, the intermediate phases of transforming a graph can be meaningful for students in perceiving function properties.

The new multiple representational software allows dynamic interactive links. Kaput (1992) introduces the notion of 'strong' dynamic interactive links, called strong links. The strong links can be explained by contrasting the old use of links in software with the new ones. The links between representations were usually made from one stage directly to the other. Recently, the software has been designed to allow continuous transformations of objects within one representation with continuous feedback in the other. Researchers called for investigations into the effect of the dynamic interactive links between different representations of function. Schwarz & Dreyfus (1993), for example, investigated students' perceptions of maximum using multiple representations (TRM). They report that the use of TRM in activities linked with the idea of maximum led the students to: recognise invariants (function properties) while creating and comparing representatives to different settings; and identify invariants while co-ordinating actions among representations pertaining to different settings.

Researchers also investigated these new dynamic possibilities applied in the Cartesian representation in order to change the way students analysed graphs. Dubisnky & Tall (1991), for example, discussed the use of Graphic Calculus (Tall et al, 1990) "to provide students with a cognitive approach" (p.238) to the concept of limit by exploring the possibility of magnifying graphs. Kieran et al (1993) reported that the interactions with the zoom associated with discrete graphs helped the students' perceptions of infinity in the sense of cardinality. Phil Lewis created the RandomGrapher with the objective of helping students to recognise a graph as a set of points, thus changing their pictorial view (Goldenberg et al, 1992). Lewis created a computer graphic generator that plots the points randomly, which creates the shape of the function randomly step-by-step. Thus, mathematics education owes a debt to technology for making it possible to change students' views when they analyse graphs.

One of the most often reported problems in research about learning function is the confusion of what is the variable, and what are the coefficients. Goldenberg (1988) and Clement (1985) suggest one reason for this is the emphasis on tasks that require students to vary the coefficients to see the transformation which has occurred in the graph instead of varying the variable. The coefficient is explored as variable and the variable as constant. On the basis of previous analysis (Goldenberg, 1988, 1991), Goldenberg at al (1992) created DynaGraph, a new dynamic visual representation where the users can vary the variable having as feedback the value of the function. As Kaput (1992) claims, "Dynamic media are the natural "home" for variables, rather than static media, which require the user to apply much of the variation cognitively" (p.534). Therefore, the dynamic manipulations in new representations can be used as tools to lead students towards a variational view of function.

In order to investigate the effect of the use of these dynamic potentials in students' perceptions of function particularly in visual representations, two programs were chosen: DynaGraph and Function Probe.

4.2 Microworlds

The potential of a computer environment can lead us to believe that interacting with it can enable students to develop their perceptions of functions. Nonetheless, researchers have shown that these improvements are not straightforward. As diSessa (1995) and Wenzelburger (1991) argue, software per se does not help students. Wenzelburger (1991, 1992), for example, showed that the possibility of graphing quickly does not in itself help students to improve their perceptions of graphic representations of functions. Students gained speed and lost involvement in the activities. The design of activities plays an essential role in facilitating students' exploration of the potential of the software. Together activities and software must compose an environment which encourages students to learn by exploring functions — a microworld.

The term microworld has been used from different viewpoints. In a technical sense, a microworld is a computer environment which embodies a concept (Papert, 1980). While analysing common characteristics of the use of a microworld, Edwards (1995) points out that in it: (a) learning is dynamic, (b) a domain of mathematics is embodied and (c) access to ideas and phenomena which are not otherwise easily encountered by the students is provided. The technical part of a microworld is a computational environment which embodies a concept, so it can be seen as representations (Edwards, 1995). The present study uses the technical part of microworlds as the embodiment of representation or multiple representation. Nonetheless, the technical view of microworlds does not entirely fulfil this meaning.

Hoyles et al (1991) call for a pedagogical approach when dealing with microworlds: "a microworld consists of software designed to be adaptable to pupils' initial conceptions together with carefully sequenced sets of activities on and off the computer..." (p.1). Thus, this approach considers that the activities, which must take into account students' previous knowledge and researcher (teacher) expectations, compose one of the main components of a microworld. Edwards (1995) summarises the functional aspect of microworlds by the actions students are expected to perform:

- "to manipulate the objects and execute the operations instantiated in the microworld, with the purpose of inducing or discovering their properties and the functioning of the system as a whole. Experimentation, hypothesis generation and testing, and open-ended exploration are encouraged";
- "to interpret feedback from these manipulations (feedback which may be provided through multiple, linked representations) in order to self-correct or "debug" his or her understanding of the domain";
- "to use the objects and operations in the microworld either to create new entities or to solve specific problems or challenges (or both)" (p.144).

With these characteristics, she claims that the activity designed for the work can play an important role in transforming a tool into a microworld. Following Edwards' (1995) viewpoint, I argue that together with activities, a software tool can be transformed into a microworld. This is the purpose of the activities I designed in Function Probe as well as in DynaGraph in this research. The set of activities and software I will call FP microworld and DG microworlds, usually abbreviated to DG or FP.

5 Research exploring DynaGraph and Function Probe

In the following section I will discuss researches on the effect of the use of the dynamic potential of DynaGraph and Function Probe (or similar programs) on students' perceptions of functions, particularly in the properties of functions.

5.1 DynaGraph

Although at first glance DynaGraph seems to be very similar to Cartesian graphs, especially in its Cartesian version, I argue that some distinctions make them two qualitatively different representations. First, in DynaGraph the variable and its image are represented separately, which does not happen in the traditional Cartesian Graphs (Goldenberg et al, 1992). Second, DynaGraph presents a function point-by-point but its motion enables the student to have a variational perception of the properties. Thus, in my view, the students can analyse the properties in either a pointwise or a variational way. Third, the "domain variable is vary-able, dynamically, by the student, clarifying its status as the variable" (op.cit.: 243). In contrast with Cartesian Graph, the shape is not the main aspect used by the students. Fourth, in DynaGraph students never see all the function at once. On the other hand, some qualitative features are supposed to be more clear such as slope, minimum and curvature.

Goldenberg et al (1992) say that they had "barely begun to investigate students' conceptions and misconceptions of function in the context of such dynamic representations" (p.235), among them DynaGraph. In fact, they investigated six pairs of mathematically successful students from 9th and 12th grade of American schooling in 40-minute session exploring functions in the parallel version of DynaGraph. From these case studies, they reported some ways in which students conceive function properties, from which I will consider those related with the properties chosen in this study.

Examining one pair working with the function $f(x)=4-3x$, Goldenberg et al (1992) show that the students can readily realise 'the direction x and $f(x)$ moves', 'the different speeds of x and $f(x)$' and the fixed points. While examining another pair exploring the function $f(x)=x^2-1$, the properties easily identified are 'the speed is not constant', 'the function has minimum value'. They point out that the students "began to refer to functions *behaviorally* in ways that were far from ... pointwise" (p.252), that were variational. Goldenberg et al (1995) argue that, in contrast, DynaGraph shows the variational well but it does not draw attention to the structure of the algorithm that computes the function. Thus, some question remains: Can the students generalise these perceptions among different functions? What are the advantages and limitations of these new perceptions? How are properties which are not linked with variation such as symmetry analysed in this dynamic representation?

In conclusion, they argue that without seeing any other representation students spontaneously involve themselves in very deep perceptions of functions that many students never even meet. Thus, the question remains how these deep perceptions can be synthesised as mathematical knowledge. That is, can the students connect these perceptions with their previous knowledge? Can they use the generalisations built in DynaGraph to generalise the corresponding property in representations previously known? Goldenberg et al (1992) mention that on using DynaGraph with numbering scales, the students return to their pointwise views of the properties. These findings, in my view, anticipate the problems students will have while trying to connect the perceptions derived from explorations in DynaGraph with mathematical school knowledge. Thus, the further question remains: Do the students change their previous way of analysing functions after using DynaGraph? Goldenberg et al (1992) mention that two of the pairs used the qualitative ideas constructed in DynaGraph to sketch a graph from a DynaGraph representation which was not yet familiar to them.

Goldenberg et al (1992) also hypothesise that the exploration of DynaGraph in a sequence from its parallel version, passing through the perpendicular one to its Cartesian version, leads students to create a logical transition from a pair of elements of $\mathbf{R}$ to a single point in $\mathbf{R}^2$. Goldenberg et al (1992) left the question: How do interactions with DynaGraph representations affect knowledge about Cartesian Graphic representation? I believe that the students have two ways of analysing the properties in the Cartesian version of DynaGraph: by a variational analysis analogous to the ones referred to by Goldenberg et al (1992) in the parallel version of DynaGraph and by analysing the behaviour of (x,y). For students who present both analyses I conjecture that either:

(a) the variational analysis will be combined with the analysis of the behaviour of (x,y). So it will allow the students to connect knowledge built in the parallel version of DynaGraph to that built in Cartesian representation. In this case its Cartesian version will be used as a bridge between the parallel version and the Cartesian system. As suggested by Goldenberg (1993), it will facilitate students' perceptions of conventions used by the Cartesian System; or

(b) the student will keep both analyses separate. As a result, this variational perception will be kept isolated in the parallel version of DynaGraph.

5.2 Function Probe

As one focus of the present study is the dynamic transformation of graphs allowed in FP, which is common to other software, I will also discuss the researches using these other software such as Algebra ToolKit.

The use of dynamic transformations of graphs has been developed by considering that:

- "proving an environment in which functions can be manipulated as entities or objects and in which the actions of evaluating and graphing are automated should help students to..." (Yerushalmy & Schwartz, 1993: 45) change their perception of functions from a procedural to a mathematical entity perception;
- seeing transformations of functions "can play a fundamental role in unifying different families of functions and in showing the invariance of transformations across these different families, since the same action that underlies a given transformation can be linked with the different visual results in a graph after a transformation is undertaken" (Borba & Confrey, 1992: 140); seeing transformations can lead students "to recognize the common impact (with local variations) of these transformations across all of the functional families studied" (Confrey, 1992a: 150). For example, Confrey (1992b) mentioned the problem of how students come to understand why horizontal and vertical stretches can be used interchangeably (but with different magnitude of stretch) on parabolas but cannot be interchanged on the step function or trigonometric function;
- providing access to researchers seeking to understand how students reason visually about shape and location when trying to fit a graph into desired points from a prototype function (Confrey, 1992a).

Thus, the questions that remain are: how do students use these transformations as a way to identify properties as variant and invariant under the transformations? For example, believing that turning point and maximum are the same concept, they can translate horizontally a parabola to investigate the changes on turning point and maximum; and how do these features modify the status of each property in the Cartesian system? From my point of view, Function Probe can be used to "provide data to suggest possible theorems" and "to seek counter-examples", as suggested by Dubinsky & Tall (1991: 231) while examining the use of computers in advanced mathematical thinking.

Confrey et al (1991b) concentrated their work on the use of contextual problems with multiple representation software (FP) in the curriculum. They showed that the students had used the translation, stretch and reflection commands to coordinate the algebraic and graphical forms of functions. They argued that the students had shifted from their perceptions of function from equation to graph, moving from a procedural view to a view of functions as a mathematical entity. On moving from process to entity, I am interested in analysing the effects of these changes on the students' perceptions of the function properties.

Schwartz & Yerushalmy (1992) used these transformations in the introduction of algebra through the notion of function to younger students. They noticed that the students used shape, detailed position and comparing graphs as a way to decide the errors in a simplification of an algebraic manipulation. They pointed out that one of the students argued that slope is far more problematic in non-linear graphs because it depends on the x-position. In my view, this result is not only important for what the student said, but for the fact that this student was able to identify the slope in non-linear graphs and to understand its dependence on the x-position. This points to a generalisation of the idea of slope from linear to non-linear graphs.

Borba (1993) concentrated his studies on exploring how students 'understand' the transformations of graphs. His work is important for me as he pointed out some changes in the students' perceptions of functions by using the transformations of graphs. He developed two case studies with a student exploring transformations of functions in different windows of FP. He pointed out that both students used the transformations as the leading method for their conclusions. Thus, this might account for the fact that most of the time they saw transformations as a process rather than as a static two step. This led the students to generalise a particular process such as $y=f(2x)$ and $y=f(3x)$ to $f(bx)$. Conversely, the start and end point of the transformations were used by these students when making the transformation in the coefficient of the equations.

The importance of dynamic transformations of graphs was also claimed by Eisenberg & Dreyfus (1994) after investigating the effects of an instructional program using Green Globs (Dugdale, 1982) with Israeli high school students' visualisation of transformation of functions. In Green Globs, transformations are allowed only as starting and ending steps. Their results showed that only simple transformations were visualised as transformed functions and only "as a sequence of two static states rather than as a dynamic process" (Eisenberg & Dreyfus, 1994: 59). The results of Borba (1993) and Eisenberg & Dreyfus (1994) led me to investigate the importance of continuous transformations, not on changing students' understanding of the transformations themselves, but on changing students' perceptions of the function properties.

Borba (1993) showed that the horizontal transformations were the ones in which his students obtained more results. He showed that the reflection was not explored very far by his subjects. He gives as the reason the fact that the reflection is not dynamic, the only dynamism on this command is the position of the reflection line. These results, once again, point to the importance of the process. A question arises from this result: is there any pattern of similarity between the perceptions derived from the exploration of each transformation?

In the case of one of his students, Borba (1993) shows that he did not experience the illusion mentioned in Goldenberg (1988) while comparing curvature on parabolas translated vertically. Borba (1993) points out that the reason for this difference could be that "working with transformations, first by visualization, then using tables and finally using algebra may have been a factor in Doug's [his subject] lack of confusion" (p.197). In my view, students can go on to create a way of measuring the curvature and realise the limitations of their previous perceptions of curvature with two parabolas translated vertically by only two steps (beginning and end).

Nonetheless, researchers continue to argue that students analyse graphs pointwisely even after exploring transformations of graphs. Borba (1993) reported that both students based their process of seeing transformations on special points such as y-intercept and turning points. Thus, the question that remains is: do students acquire any other way of analysing the properties in graphs by exploring the transformations?

6 The research questions in the context of the software programs

After describing DynaGraph and Function Probe and the research with these software programs, the research questions are re-written to take account of the context of each of the environments. In DynaGraph, the questions are operationalised as:

Q1a: How does the interaction with the dynamic way of representing function in DynaGraph lead the students to perceive the different properties?

Q1b: What are the limitations and advantages of the perceptions built in DG microworlds?

Q1c: How do students' perceptions of the properties change from DG Parallel to DG Cartesian?

Q4a: How does the sequence from DG Parallel to DG Cartesian contribute to students' perceptions of the properties in the Cartesian representation?

Q4b: Does DG Cartesian work as a bridge for synthesis?

Q5a: Do students change their previous way of analysing functions after working with DG microworlds? If so, how? If not, why not?

The research questions specific to FP microworld are:

Q1d: How do students use the transformations of graphs to discriminate and generalise the properties?

Q1e: Are there patterns of similarities between the commands and the change in students' perceptions of the properties?

Q1f: How does exploration of dynamic transformations of graphs affect students' perceptions of the function properties?

Q5b: What are the effects of the interaction with dynamic transformations of graphs on students' knowledge of graphs?

The research tries to answer questions regarding connections made between the different microworlds:

Q4c: How does this synthesis take place?

Q4d: Which mechanisms of synthesis are suggested by the synthesis students will make?

IV — Methodology

An empirical study was designed to investigate students' perceptions of the function properties while interacting with the dynamic microworlds. The outline of the empirical study will be the first section of this chapter. Then, its design will be presented and justified in four sections: the choice of the subjects, the investigation of the students' previous knowledge and school approach to function, the main activities called research environment, and the investigation of synthesis when it is motivated. Then, the methodology of data collection during the research environment will be presented. Finally, the methodology of the analysis will be discussed. The final design for this empirical study was obtained from two previous ones: a pre-pilot with one pair of students and a pilot study with three pairs of students. The findings of the second of these will be summarised in chapter V.

1 Outline of the empirical study

This research comprises case studies undertaken in Brazil with four pairs of students from the second grade[1] of secondary school working through a sequence of tasks using three different microworlds: DG Parallel, DG Cartesian and FP. Each pair of students participated in thirteen sessions: one session for a questionnaire to characterise the students and the pre-test, one session for familiarisation in the research environment, five sessions for activities in FP, five sessions for activities in DG, and one session for the final interview. The students, from two different attainment levels, followed the activities in two different sequences: two pairs did the activities in both DG Parallel and DG Cartesian followed by the activities in FP, and the other two pairs followed the activities in the opposite order. The two different sequences were also designed to allow the analysis of the influences of students' perceptions derived from one microworld on the perceptions derived from the other. Diagram 1.1 shows the flow of the activities carried out by each pair of students in the empirical study.

As this research takes into consideration students' previous knowledge, the following was undertaken: a test of previous knowledge of functions and an interview with

[1] The second grade of Brazilian secondary schooling can be seen as corresponding to the twelfth year of English schooling using age equivalence and considering an ideal Brazilian student who did not fail in any of previous grades.

their mathematics teacher. Additionally, the curriculum materials used by the students were collected and analysed.

Diagram 1.1

Flowchart of the activities of the empirical study

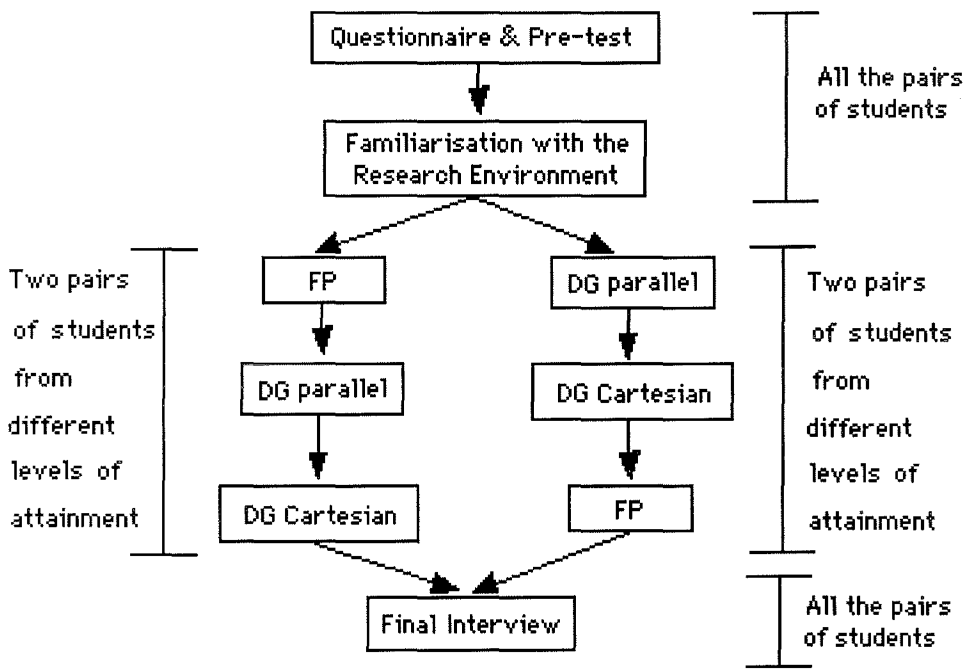


Diagram 1.2

Composition of the research environment

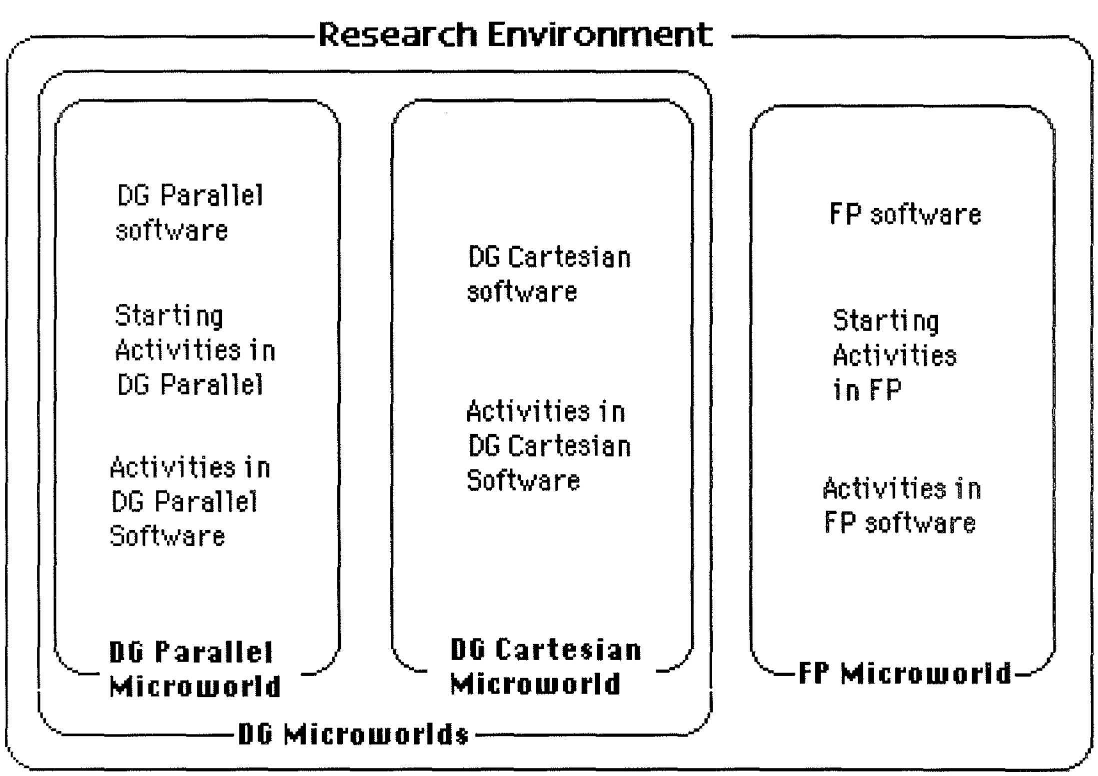


Diagram 1.2 shows that the research environment consisted of three microworlds. Each microworld was essentially built with activities designed to create an exploratory learning environment around each software. In each microworld, the activities had the following structure:

- The first phase was intended to be a session to familiarise the students with the software commands. This phase was not included in DG Cartesian.
- In the second phase, the students were asked to describe a set of functions corresponding to the following: $y=6$, $y=-3$, $y=x$, $y=-x$, $y=2x$, $y=x-6$, $y=0.25x^2$, $y=-0.25x^2$, $y=0.5x^2$, $y=0.25x^2-8$, $y=7\sin(0.25\pi x)$, $y=7\sin(0.125\pi x)$. They were required to characterise and distinguish these functions.
- In the last phase, the students were required to group the functions according to the properties they had observed.

Finally, the students were interviewed to verify whether their perceptions derived from activities in one microworld were connected to their previous knowledge or to their perceptions derived from activities in the other microworlds.

2 The case study students

2.1 Choice of school

The Brazilian educational system, nowadays, has two kinds of secondary school according to their purpose: academic schools which attempt to prepare students for higher education, and technical schools which prepare students for technical jobs. I addressed my study to the first type of school, which, according to Werebe (1994), represents the majority of secondary schools in Brazil. At these schools, formal mathematical knowledge is the main preoccupation of maths teaching.

All students belonged to the same class in the same secondary school in Brazil. Two criteria influenced my choice of school to work with. The first criterion was access to information in the school and the teacher's availability and willingness to carry out required tasks. The second criterion was that the school should not be an exception to the way mathematics is normally approached in academic secondary schools in Brazil. Taking both criteria into consideration, I chose a federal[2] state school. The support I received for my research from this school greatly facilitated the development of this study.

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[2] The federal schools are state schools which belong to the Brazilian central government.

Looking at Brazilian state schools as they really are, I might say here that this is not an example of an average state secondary school in Brazil. This school can be considered as a model of excellence among state schools in Brazilian education as it actually is. It does, however, take the standard view of mathematics followed by academic secondary schools. I did not try to find a typical state school because the focus of my study was not on the social aspects of Brazilian education. To consider these problems a sociological approach would have been necessary to the research. As Werebe (1994) points out, there are many reasons for the failure of Brazilian state education such as: the majority of students spend forty hours a week in outside jobs, the teachers are very badly paid, there are great incentives for 'private'[3] schools as opposed to state schools, and state schools consequently have depreciated.

2.2 Students

The eight students were chosen from the second year the federal state school, when the students have already studied the topic of function. By selecting students from one grade only, I was unable to select them by age. In Brazil, although the elementary school is composed of 8 grades (with one year each), according to Wilbie (1992) it takes on average 11.8 years for students to complete it. In this context, I tried to limit the range of ages to between 15 and 18.

The students were chosen from a group of volunteers. After explaining to the class the nature of this study, their teacher asked for volunteers to participate in the research. The teacher classified the volunteers by attainment levels. Then, he and I chose and grouped the pairs taking into consideration other criteria, which I discuss below.

The students were selected all from the same class to take into account their previous school knowledge of function and its influence on the way they would approach the activities in this investigation. The students were selected from different attainment levels in order to provide the analysis of a variety of students. The mathematics teacher had allotted all the students to three attainment levels: the lower (LA)[4], the middle (MA)[5] and the higher (HA)[6] attainment levels.

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[3] In Brazil, 'private' schools are those which belong to an individual person, institution, or church. cf. Public schools in England. The term 'public' was not used here because in Brazil the term 'public' would correspond to 'state' schools in England.

[4] The lower attainment level comprised the students who usually needed extra help to succeed in school mathematics exams.

[5] The middle attainment level comprised the students who sometimes need extra help to succeed in these exams.

[6] The higher attainment level comprised the students who have no difficulties in succeeding in these exams.

Certain constraints led me to choose the students for the investigation from MA and LA levels only. First, as I had to observe each pair in turn, the constraint in their availability[7] led me choose to work with four pairs of students which was sufficient for the investigation. Second, the research was designed to investigate two pairs of students from each attainment level. Therefore, I had to select students from two attainment levels. Finally, I omitted students from HA because this study is based on what students say or write about function while carrying out the activities. The possibility of these students doing all the work without discussing it would invalidate the investigation.

The four students chosen from each attainment level were grouped in pairs. Therefore, each pair of students was homogeneous according to the attainment level in school mathematics, the aim being to reduce the likelihood of the dominance of one student. The experience of the pilot study led me to introduce two new criteria in the choice of each pair of students: the students had shown no previous antipathy to their partner; and if they had worked in groups before, their behaviour in these groups was taken into account. For example, I avoided assigning to the same pair two students who had presented dominant/passive behaviour.

In order to describe the students one questionnaire answered by the students and one interview undertaken with the mathematics teacher were undertaken. The questionnaire (see section AII-1) aimed to obtain students' personal information and to characterise their interest in mathematics and computers. One of the purposes of the interview (see subsection AII-2.1) was to investigate the criteria used by the teacher in assigning each student to each attainment level. Both interview and questionnaire let me to give some characteristics the students.

3 The students' previous knowledge

3.1 The school approach to functions

In the search for similarities and differences between students' barriers while exploring the microworld and the school approach to functions, it was important to examine two points: the way the students learn about function at school and the role of the topic of function in the academic secondary schools of Brazil. An interview with the mathematics teacher was undertaken, from which the

[7] The timetable limited the number of pairs of students I was able to work with. Being from the same class, all the students were available to work at the same times during the weeks in question.

curriculum materials used by the teacher to explore functions were collected particularly the ones used with these students. Among the material collected, the following were included: a list of topics explored in the mathematics curriculum, the textbook used by these students, other materials used to cover this subject, and a sample of written work produced by the students such as their notebooks.

The interview with the teacher was also designed to allow me to understand how he used the curriculum material with these students and to map:
- The sort of activities developed in their mathematics classes;
- The work done prior to the introduction of functions;
- The introduction of the topic of function;
- The activities carried out when developing the topic of function;
- The exercises given to these students exploring the concept of function.

It was also organised in order to clarify the following points in the teacher's exploration of functions with these students:
- The role he attributed to the definitions;
- The representations used to explore function;
- The activities he carried out in each of the representations;
- The properties he emphasised while exploring functions.

3.2 Pre-test

A test was designed to access the students' previous knowledge of function (see section AII-3). The analysis of this pre-test will be the starting point of a longitudinal analysis of students' perceptions of the properties. Considering that this research focuses attention on 'how' the students perceive the selected properties, open questions were chosen instead of multiple choice ones. Open questions allowed me to access the arguments used by the students while exploring the properties and also revealed different perceptions about the same properties of function.

The complete pre-test included seventeen questions of three types:
- those about the meaning of mathematical terms;
- those about interpreting information through graphic and/or algebraic representations;
- those to test other mathematical skills in these representations.

The first type of question, which includes 2, 4, 5, 9, 11, 12, 13 and 17 had the following form: what do you understand by ...? They were introduced as a result of the analysis of the pilot-study data. When the students failed in one question involving one property, I was not able to say whether they did not understand the property or the term I used to denote it.

The second type of question includes 1, 7, 10, 16. Questions 1 and 16, which were designed by myself, requested the students to interpret the same information: one (first) from equations and the other (sixteenth) from graphs. With them, I aimed to compare the students' interpretation of the properties: derivative, second derivative, and meeting point in graphs and in equations. Question 7 was to interpret the properties: extreme values, monotonicity and derivative from a graph mainly constituted by points. In this case, the interpretation of graphs has been pointed out to be more easily done by students (Goldenberg, 1988) than in the case of differentiable graphs. This question was adapted from a question in Iezzi et al (1990: 55), the textbook used by the school. It was modified to give 'sense' of the lines which link the points. Question 10 required the students to interpret the properties of monotonicity and range in a differentiable function from graph and equation. The question was introduced in the test because the function represents counter-examples of *associations*[8] developed by the students from the pilot study. It also investigated whether students extrapolate graphs.

The other questions (3, 6, 8, 14 and 15) examined students' skills while investigating their perceptions of function properties. Question 3 requested the students to compare the curvature of four parabolas. It was included in the pre-test because it is one of students' difficulties in graphs that are pointed out as being a 'misconception' (Dreyfus & Eisenberg, 1982 & Goldenberg, 1988). This question was adapted from a question by Dreyfus & Eisenberg (1982: 192). Question 6 requested students to sketch seven graphs from constant, linear, quadratic and trigonometric equations. It intended to access: how the students trace or sketch graphs; if they compare equations from the same family; how able they are in plotting points. This question also required the students to identify: the periodic functions, the functions with bounded range and the turning points. The construction of graphs from verbal description was explored in question 8, which was created by myself. The students were requested to sketch a graph of distance per time which represents the motion of a car. My intention was to investigate students' perceptions of different properties related to variation such as: constant and variable speed; straight lines and curves; motion and motionlessness; and horizontal straight lines. Questions 14 and 15 were created by myself to verify *associations* presented during the pilot study. In question 14, the students were asked to identify range and extreme

8 'Associations' was defined as students' perceptions of a property which is connected to a different property or at least limited to special cases of functions.

values in five graphs. In question 15, they were asked to identify periodic and symmetric graphs. With this question I intended to verify: whether the students distinguish a periodic graph from any oscillatory one, and how they identify symmetric functions and line of symmetry.

4 The research environment

The research environment comprised three microworlds, each designed with activities around one software program: DG Parallel, DG Cartesian and FP. FP program has already been described in chapter II. Both DG programs were adapted from DynaGraph to fit the requirements of this research. In subsection 4.2 both adaptations of DynaGraph will be described and justified. Subsection 4.3 will justify the choice of FP and DynaGraph. The designed activities will be described and justified in the subsequent subsections.

4.1 Familiarisation with the research environment

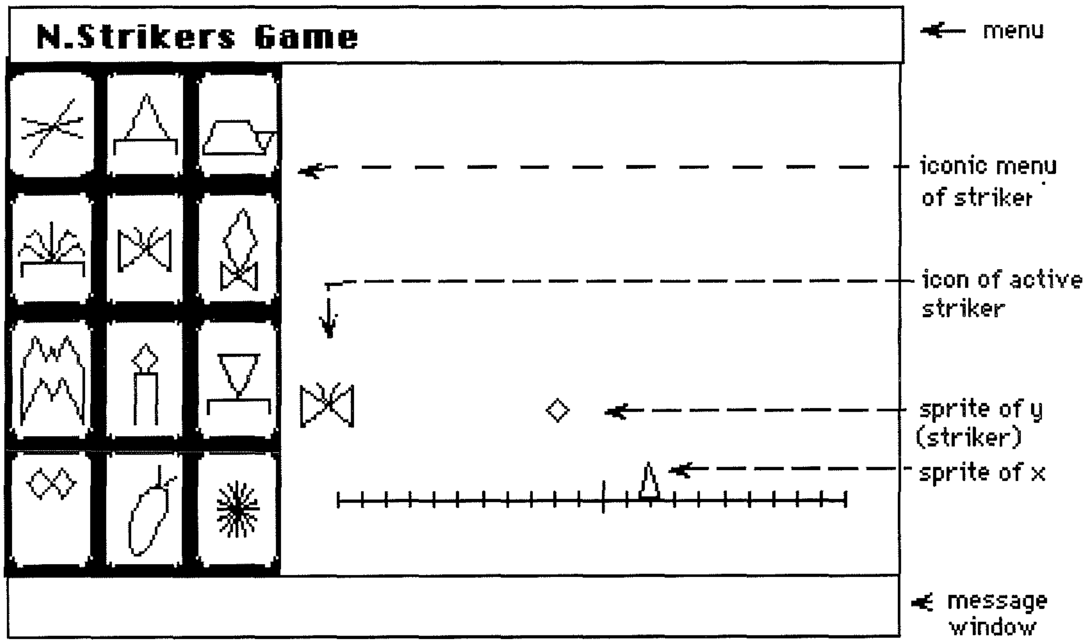
The students participated in one session designed simply to familiarise them with the research environment. This session took place before the ones for the research environment, using an adaptation of a Journey across Function Probe software called Pizza (see section AI-4) from Confrey et al (1991a). As (a) the instruments used to collect data (such as video-camera, tape-recorder, and notepads) interfered with students' behaviour by inhibiting discussion and (b) this interference was marked in the first session and tended to disappear in the following ones, the familiarisation session was valuable in avoiding the first research environment session from been wasted as a source of data. Secondly, in the pilot study, I noticed that to operate Function Probe the students needed more than one session, thus, FP was also used in this session.

4.2 DG Parallel and DG Cartesian programs

DynaGraph's way of representing a function was adapted to another environment, which I will call DynaGraph Game (DG). The term 'Game' was originated in the first version of this environment used in the first activity with DG Parallel (see section AII-1) which is a computer-game with the same structure as DG Parallel. Here, I will describe two versions of DG (Parallel and Cartesian) without the game features. DG explores functions as behaviour of strikers using the same representational system of DynaGraph. Figure 4.1 shows the screen of DG Parallel with the function of y=-x displayed.

Figure 4.1

Screen of DG Parallel with the striker of y=-x displayed

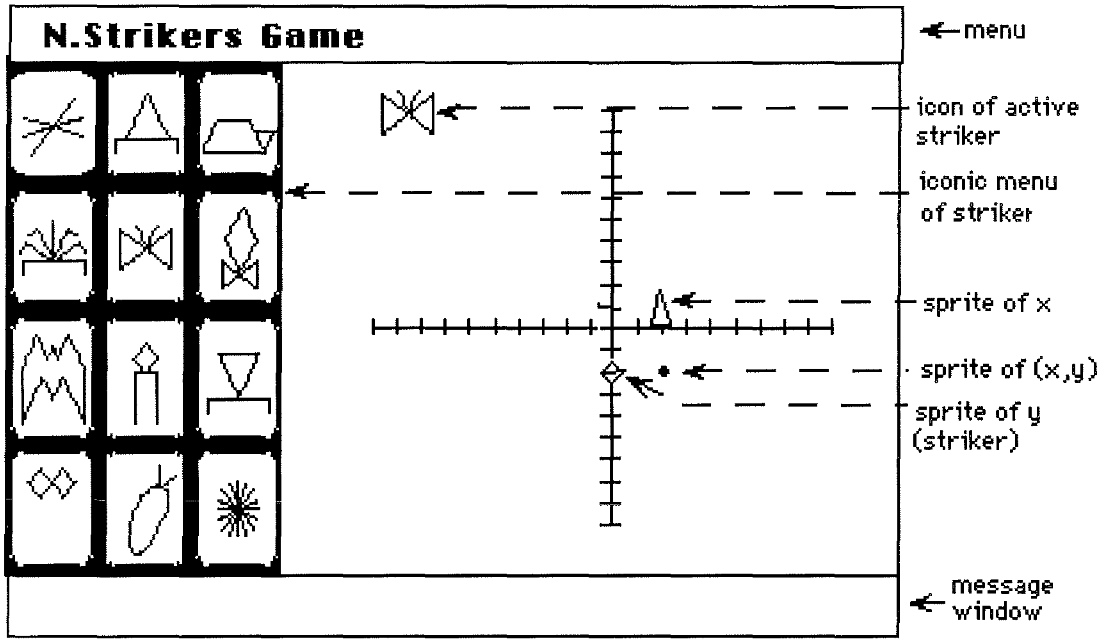


DG Parallel explores the twelve functions listed in section 1 hidden in strikers which are presented by different icons. By clicking the mouse on the iconic menu, users can change the active striker (active function). By displaying the icon of the active striker, DG Parallel enables students to remember which 'function' is on. The representation of x and y is done in the same way as DynaGraph. When pulling x to the left or right sides with the mouse, students receive as feedback the variation of y according to the function hidden in the active striker. In DG Parallel, 'the step x moves' was fixed at 0.5. The scales of x and y are the same. In DG Parallel students can choose to observe up to three strikers at the same time. If more than one striker is chosen, the strikers are displayed in parallel lines. In this case, if students decide to change one of the active strikers, a message requesting them to identify which one they want to replace is displayed at the message window.

DG menu has two options: Number of Strikers and Game. The Game option has three items: Start, Stop, and Quit. These items mean: start the program, stop the current choice of the strikers and quit DG. The 'N.Strikers' option has three items: 1 striker, 2 strikers and 3 strikers. More than one striker is usually chosen to compare the behaviour of different strikers.

Figure 4.2

Screen of DG Cartesian with the striker of y=-x displayed



DG Cartesian has the same characteristic as DG Parallel. The same functions are hidden in the same strikers. It differs from DG Parallel in that:

- the axes appear as in the Cartesian system;
- a dot representing (x,y) is added;
- when more than one striker is used, the active ones are discriminated by colours.

In both DG Parallel and DG Cartesian, students explore DynaGraph's dynamic ways of representing the twelve functions without having access to any other representation of them, in particular any algebraic representation. This was the main reason to produce the adaptation, instead of using DynaGraph directly. The 'behaviour' option from DynaGraph was not used here because it does not make clear which behaviour (function) is active. The use of icons helps students to match behaviour and strikers. Another reason is that by using an iconic menu students are allowed to easily change the active striker easily whenever they want to.

4.3 Rationale for the choice of the programs

In conventional multiple representational software, the Cartesian representation is used only as a feedback window. In FP, the real possibility of manipulating visual representation offered by the computer has changed the 'face' of the multiple representational software — transformations of functions are no longer a privilege of the algebraic or the tabular representations. FP allows dynamic transformations of graphs in the Cartesian representation. All the transformations can be operated inside the Cartesian representation with the change in the equation as feedback.

In the choice of DynaGraph (Goldenberg et al, 1992), I consider that the opportunity to manipulate x and see how y varies allows the students a completely different perception of properties related to variation (increasing, constancy, speed, turning point). In addition, changing x and seeing the changes in y and (x,y) in the Cartesian version of DynaGraph can enable the students to develop a variational view in the Cartesian representation of these ideas.

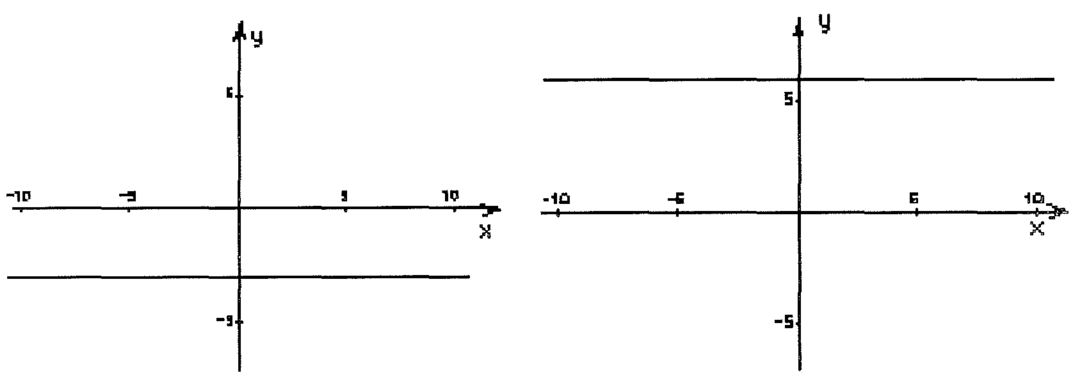
I believe that both programs allow students to explore the properties of function using visual dynamic representations. Moreover, they represent an opportunity to shift the emphasis from algebraic to the visual representation of functions. A parallel between the use of these programs is that: the dynamic transformations of graphs offered by Function Probe allow students to observe function properties by the variation of these properties, while the dynamic way of representing a function in DynaGraph allows students to observe the function properties by varying the variable.

4.4 The choice of the functions

The choice of the sample of functions to be used played an essential role in the construction of the microworlds. The following twelve functions were chosen:

Figure 4.3

Graphs and equations of two of the chosen functions

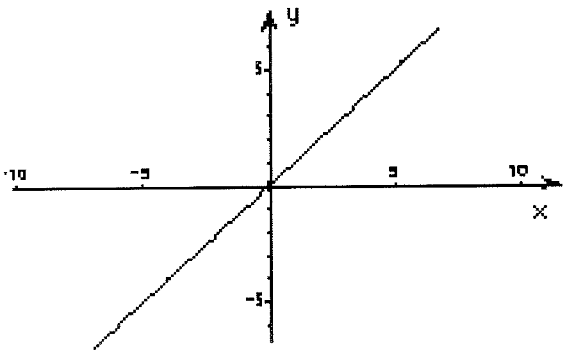


y = -3

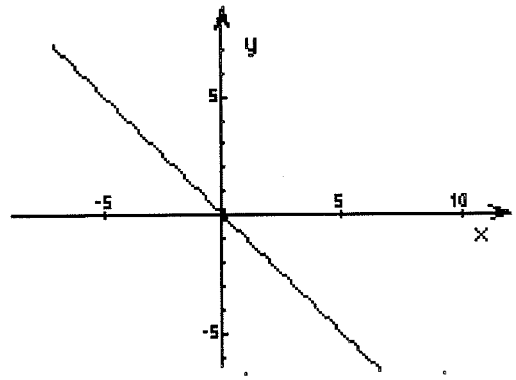
y = 6

Figure 4.4

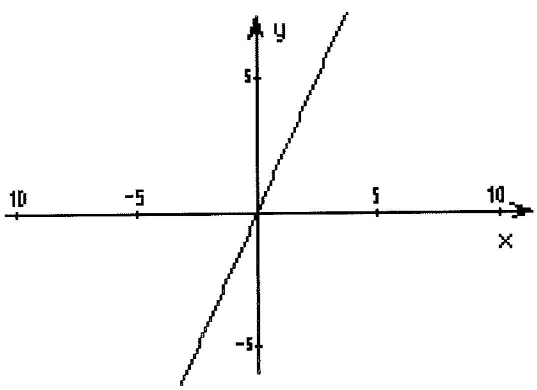
Graphs and equations of six of the chosen functions



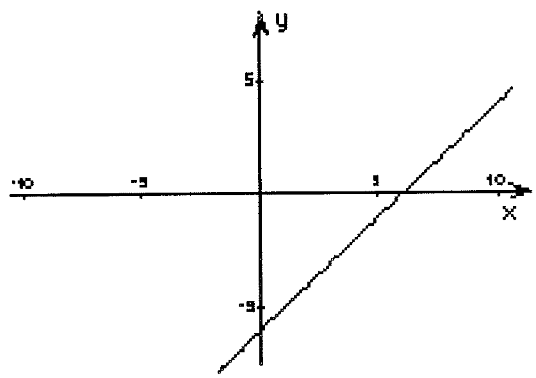
$y = x$



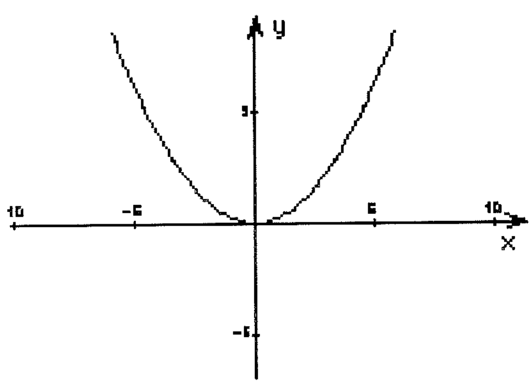
$y = -x$



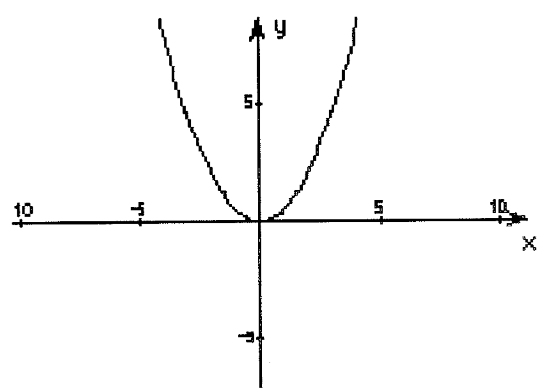
$y = 2x$



$y = x-6$



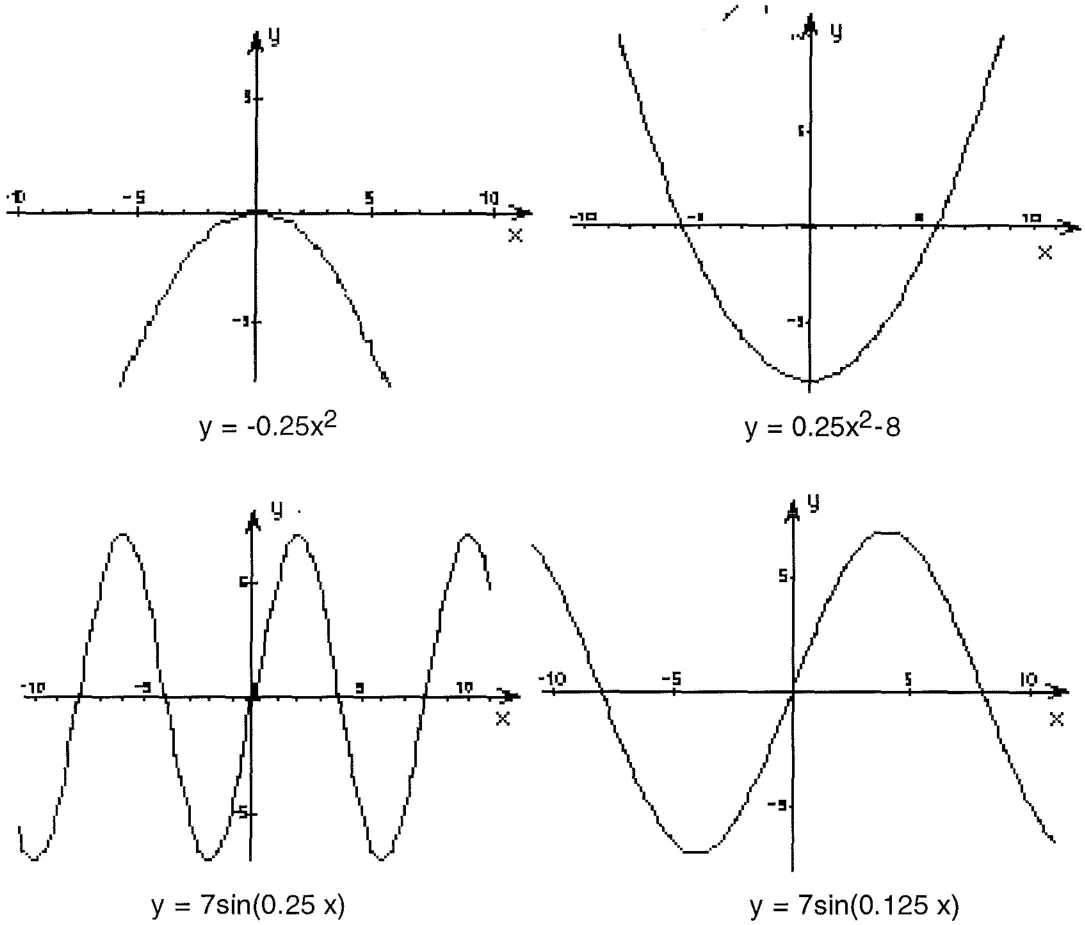
$y = 0.25x^2$



$y = 0.5x^2$

Figure 4.5

Graphs and equations of the other four functions chosen



The selection of these functions was a result analysis of data from two pilot studies. The number of functions was defined after the pre-pilot study which had started with twenty-one functions. In order to choose the functions two criteria were considered: the properties had to be emphasised by the sample and the dynamic potential of FP and DynaGraph could be used when exploring the functions.

Firstly, the chosen functions are linked with the dynamic transformations of graphs allowed by FP because they all belong to four families of functions: constant, linear, quadratic, and trigonometric. Each of these graphs can be dynamically transformed into another from the same family or into the graph of a constant function. Also, graphs belonging to different families cannot be transformed into each other, apart from the constant one. Thus, the students can explore the function properties while trying to transform a graph into another. Within a family, each graph can be obtained with only one transformation of the prototype function of each family: y=6, y=x, $y=0.25x^2$ and y=7sin(0.25 x). One transformation alters some of the chosen properties keeping the others invariable. For example, on changing the graph of

$y=0.25x^2$ into the graph of $y=0.5x^2$, the curvature and slope of the first parabola will be modified but its line of symmetry, range, turning point and domains of monotonicity will be maintained. On the other hand, the transformation of the graph of $y=0.25x^2$ into the graph of $y=0.25x^2-8$ will vary the turning point and range only.

Secondly, DynaGraph was important in defining the families used as well as the adjustment of the coefficients in the equations. The families were chosen in order to exploit the dynamic way DynaGraph represents a function. For example, I tried to make clear the difference between the behaviour of functions with constant speed (linear and constant ones) and functions with variable speed (quadratic and trigonometric ones). After the functions were defined into families, the choice of the coefficients tried to emphasise differences of the same properties for different functions in DynaGraph. For example, on trying to make clear the difference between constant and variable speed, I had to choose the coefficients to highlight these differences in domain which would be visible on the screen. This is why the quadratic and trigonometric functions present such unusual coefficient. Also, within each family, the coefficient choice had to make clear properties which vary within a family. For example, different speed (derivative) of different linear functions had to be clear.

Table 4.1 presents the properties emphasised by the similarities and contrasts within and between families of functions. The cells in diagonal refer to emphasis within a family. The cells above the diagonal refer to the emphasis due to contrasts between families of functions while the ones below refer to the emphasis produced by similarities between families.

Table 4.1

Similarities and contrasts of the properties within and between the function families

FAMILY OF FUNCTION	Constant	Linear	Quadratic	Trigonometric
Constant	Different ranges Constant deriv. Limited range	Limit. x unlimit. range Derivative	Limit. x unlimit. range	Const. x variab. derivative
Linear	Constant deriv.	Different deriv. Constant deriv. Monotonicity Unlimited range	Const. x variab. derivative Chang. x Maint. monotonicity Limit of range	Const. x variab. derivative Limit. x unlimit. range Monotonicity
Quadratic	Limit of range Line symmetry	Unlimited range Derivative different from zero	Second derivat. Turning points Line symmetry Variable deriv. Limit of range Domain of monotonicity	Limit. x unlimit. range Domain of monotonicity
Trigonometric	Limited range Line symmetry	Derivative different from zero	Turning points Limit of range Variable deriv. Line symmetry	Period Periodic Turning points Limited range Same range Line symmetry

4.5 Activities of the study

The activities played the role of transforming the software into an exploratory environment for functions, into the microworlds. They were designed to lead students to:

- explore the properties of the twelve functions chosen into four families (constant, linear, quadratic and sine functions);
- discuss these properties between themselves.

The two sets of activities all had the following features:

(a) the students would work in pairs;

(b) the activities would be composed of description and classification of the functions in each of the computer programs;

(c) the descriptions would always be made in a describing/guessing activity, in which one student was expected to guess what was the function described by his/her partner.

The features (a) and (c) aim to promote students' discussions about the function properties. As a student-centred reasoning study and considering that through language the students articulate their thoughts and communicate and negotiate a common perception (Hoyles & Sutherland, 1989), this research used small group-work in a case study. To choose the number of students in a group I took into consideration Hoyles & Sutherland's (1989) case study using Logo, in which two students and the computer feedback interact well. In each setting of activities, I expected students to describe and classify the twelve functions.

By considering that on generating mathematics, language is one of the most important points, the activities must have a balance of interaction with the computer and descriptions (Noss & Hoyles, 1996). Confrey et al (1991b) argue that in describing and classifying students try to examine and search for invariants. In addition, Goldenberg et al (1992) showed that when classifying functions, students discuss and reflect on the behaviour of function as well as comparing the behaviour of different functions. Thus, I aimed to lead the students into exploration of the function properties by ask them to describe and classify them according to their representations: as graph in FP and as behaviour of strikers in both DG microworlds.

In using a description/guessing activity, my aim was that each student should:

- try to understand his/her partner's descriptions of a function;
- look for properties his/her partner used to characterise each function;
- compare the description of a function given by his/her partner to his/her own perceptions of the function properties;
- discuss the accuracy of a description when it can be fitted to more than one function or none of them;
- compare different functions by trying to match a description with the twelve functions;
- search in different functions for properties previously observed in one of them;
- negotiate and complete each other's descriptions.

These actions would lead the students to:

- discover new properties for characterising each function;
- revise their perceptions of the function properties;
- generalise their perceptions of one property to a wide range of functions.

In using classification activities, I intended to lead students to:

- search for variants and invariants of the different functions;
- negotiate a common classification by discussing their perceptions and their language;
- compare the properties within and between different families of functions;
- develop arguments for grouping the functions.

In my opinion, this classification can help students to generalise their perceptions to a wide range of functions. Also, the arguments used by the students during the classification would reflect the main features observed by the students and/or features from their previous knowledge.

The activities were designed to take place over ten sessions: five for FP, three for DG Parallel and two for DG Cartesian. The first sessions with FP and with DG Parallel were created to familiarise students with the microworld. In DG Cartesian, no session for familiarisation was necessary because its use is very similar to DG Parallel. Tables 4.2 and 4.3 summarise the activities in each microworld. A detailed description of the activities is presented in appendix II and the material used such as worksheets and cards in appendix I.

The sessions occurred twice a week for each pair of students. Almost all the sessions were designed to take on average two hours, but this duration was flexible. Considering the natural differences in student's development, fixing the duration of each session would make sense only in order to compare students' performance.

Table 4.2

Activities of DG microworlds

Micro world	Sessions	Duration (average)	Activities	Material
DG Parallel	First - Starting activities	1 hr. 30 min.	(1) Play with the strikers in DG Game	DG Game
	Second - Description	2 hr.	(2) Describe the behaviour of strikers for the partner to guess it (in sets of 2 or 3 strikers); (3) Guess which strikers were described	DG Parallel software Worksheet 12
	Third - Classification	2 hr.	(4) Group the behaviour of striker according to their descriptions	DG Parallel software Descriptions done A3-paper 12 cards, each with one of the icons of the strikers Worksheet 13
DG Cartesian	First - Description	2 hr.	(1) Describe the behaviour of strikers for the partner to guess it (in sets of 2 strikers); (2) Guess which strikers were described	DG Parallel software Worksheet 12
	Second - Classification	2 hr.	(3) Group the behaviour of striker according to their descriptions	DG Parallel software Descriptions they wrote A3-paper 12 cards, each with one of the icons of strikers Worksheet 13

Table 4.3

Activities of FP microworld

Sessions	Duration (average)	Activities	Material
First - Starting Activities	1 hr. 30 min.	(1) Transform the graph of y=abs(x) using one of the commands (2) Describe the transformed graph for the partner to guess it (3) Guess the transformed graph	FP software (Only graph window on) Worksheet 1 Worksheet 2
Second, third and fourth - Description	2 hr.	Describe graphs for the partner to guess in two different ways: (1a) Choose two functions using the equations, and trace the graph of one of them, (2a) Try to obtain the other equation transforming the graph of the first and describe one of the graphs by comparing the two graphs. (1b) Choose one function using the equations (2b) Describe the chosen graph after exploring all the commands on it (3) Guess: the obtained function in the case (a) or the chosen function in the case (b) using the cards with graphs only	FP Software (graph window only) 12 Cards, each with one of the equations Worksheet 3 (a) Worksheet 4 (b) 12 Cards, each with one of the graphs of the functions
Fifth - Classification	2 hr.	(1) Classify the graphs into groups; (2) Choose one function of each group to explore the commands on it. (3) Describe variants and invariants of the graphs of each group	FP Software (graph window only) A3 paper Worksheets 5 to 10 12 Cards, each with one of the graphs

5 The final interview

The main aim of the final interview was to investigate how far the students were able to connect perceptions built within one microworld with their previous knowledge and/or with those built within other microworlds. As in the pilot study many of these perceptions were not spontaneously connected with knowledge from other microworlds, I was not sure whether the students were not able to connect or whether they did not clearly express the connections. Thus, the final interview was introduced to complete the analysis of the synthesis students can achieve while exploring the dynamic potential of the microworlds in the research environment.

Table 5.1

Detail of activities developed during the final interview

Act. No.	Mater. avail.	Activities	Justification
(a)	DG Parallel 12 Cards, each with one of the graphs 12 Cards, each with one of the strikers A3-paper	The students are asked to match the behaviour of the strikers with the graphs, using the cards of graphs and exploring the strikers in DG Parallel. The pairs of strikers and corresponding graphs are placed on the A3-paper. The researcher encourages the students to discuss the criteria they are using to build each pair. The researcher asks them why the striker and the graph of each pair match.	Lead the students to connect properties from graphs to properties from strikers by investigating criteria to match them.
(b)	DG Parallel The strikers and the graphs matched by the pair of students	For each perception built within DG microworlds: The researcher shows the students the behaviour of the strikers, reminding them of the perception they built. Then, the researcher asks the students to identify the corresponding characteristics in the cards of the graphs. The researcher asks the students how they know the correspondence.	Investigate if and how the students can identify the perceptions built within DG microworlds in graphs already matched with the strikers.
(c)	DynaGraph FP	For each perception built within FP microworld: The researcher asks the students to identify it in the behaviour of the strikers, reminding them for which graphs they built it. The researcher shows the transformations which make the property invariant and those which change it, asking the students to predict the change in the behaviour of the striker corresponding to each graph. The researcher allows the students to compare the behaviour of the two strikers in DynaGraph to verify their predictions	Lead the students to identify the properties from graphs to strikers by their variance and invariance under the transformations of graphs. Lead the students to connect the properties they had observed changing in FP to the properties which change in the behaviour of the strikers. Investigate how the students perceive the properties in the behaviour of the strikers.

The final interview had three stages in which the students were asked to:

(a) match the strikers with the graphs;

(b) identify perceptions built within DG microworlds in the graphs;

(c) predict the behaviour of a new striker which corresponded to a graph transformed from another using FP, having the behaviour of the striker corresponding to the graph.

All these phases were designed from the analysis of the pilot study data. They constituted points at which these students made spontaneous connections. The phases of the final interview are detailed in table 5.1 which also contains their justifications.

The final interview investigated only the perceptions actually expressed by each pair of students. Thus, the questions for the final interview were different for each pair of students and could not be previously written. Nonetheless, a draft of the final interview was designed with prototypes of the questions for activities (b) and (c) and notes of the perceptions built by the students during each session were taken in two notepads (see subsections AI-8.2 and AI-8.3). The prototypes of the questions together with the students' perceptions composed the final interview.

6 Data collection

The data of this research were collected by video-tape records of the sessions, notes taken by the researcher, the questionnaires and worksheets filled in by the students, transcriptions of the interviews and the collection of curriculum material. In this section, I will discuss the role of the researcher while observing and interviewing and that of the notepads created to facilitate note-taking.

6.1 The role of the researcher

Although this research was composed of participant case studies, the interference of the researcher during the sessions was restricted according to the goal of the intervention. Since the general goal of this study was to analyse the students' arguments while describing and classifying functions, intervention by the researcher giving mathematical teaching would be inappropriate as it could interfere the students' arguments and classification. The researcher only intervened for the following purposes: to explain the activities and the computer commands; to stimulate the students' discussions, to investigate the students' thoughts, to understand the students' language, and to understand on which representation the students were focusing their arguments.

In the final interview, the interventions of the researcher aimed to obtain information about the connections the students were building between properties in different microworlds. The 'why' questions were used to investigate the properties the students were using when matching strikers with graphs, for example, 'why did you match striker A with graph B?' or 'why do you think they are similar?'. The 'how' questions aimed to allow the researcher to go deeper in understanding the connections the students built. An illustration of these questions is 'how do you know that property A in graphs corresponds to property B in strikers?'.

In the other interviews of this study, the researcher was allowed to vary a question when noticing that the question was mis-interpreted.

6.2 Observational tools

All the sessions were recorded with a video-camera focusing on the computer screen. The observations of the pilot study revealed that in the sessions of classification the A3-paper, where the students grouped the cards of the functions, was another focus of actions. As only one video-camera was available, the researcher took notes of the functions grouped at each moment.

Other sources of data were worksheets and notes taken during the sessions. The pre-pilot study showed that the researcher needed an easy way to take notes while observing the sessions. Thus, two notepads, one for DG microworlds (see subsection AI-8.1) and one for FP microworld (see subsection AI-8.2), were designed according to the characteristics of each microworld. In the one for DG, the menu of strikers enabled the researcher to identify (by ticking) the striker(s) to which the notes referred. The one for FP presented the equations of the functions and a menu with the icons of 'transformations' commands to help identify the function(s) and transformation to which the notes referred. After the pilot study, two other notepads (see subsections AI-8.3 and AI-8.4) were designed to enable the researcher to build the final interview before analysing the data. These notepads presented a list of perceptions built by the students who participated in the pilot study and blank spaces to help the researcher identify the ones built by the pair of students in each microworld. The researcher had to tick the perceptions built during each session and write beside them the functions for which they were observed.

7 Overview of strategies of analysis

The analysis of the main study had three phases:

- Analysis of the school curriculum on the topic of function;
- A longitudinal analysis of the work of each pair of students;
- A cross-sectional analysis of the work of the pairs.

7.1 Analysis of the school curriculum

The analysis of the approach used by the teacher to introduce the topic of function to the students has as source the curriculum material and the interview with the mathematics teacher.

This analysis aims to give information on similar patterns between the perceptions the students developed in the empirical study and the way they were introduced to the topic. Thus, the analysis of the curriculum material and the interview with the teacher focused on the work these students did before being introduced to functions as well as during the topic itself. I consider: how the chosen properties were used in the topic; for which family of functions they were explored; the representations explored and how they were explored. These points will lead me to predict over-generalisations and knowledge-obstacles the students might exhibit during the empirical study.

7.2 Longitudinal analysis of the work of the pairs of students

The longitudinal analysis of the work of each pair of students has as sources: the individual student's tests of previous knowledge; transcriptions of the discussions during the sessions; the material written by the students during the activities; the video-tape records of the sessions; the researcher's notes; and in the case of FP the computer records.

The longitudinal analysis examines the development of students' perceptions of the function properties. First, a summary of students' previous knowledge is made from the analysis of their pre-test. Second, the students' perceptions of the property constructed during their interactions in the research environment is examined. In this part of the analysis, I considered the usefulness, limitations, origin of these perceptions as well as how and when the students came to discriminate, generalise, associate, and spontaneously synthesise these properties. Finally, the analysis of the connections motivated in the final interview is presented.

The longitudinal analysis is undertaken property by property. For each property of function, I looked for:

- the influence of the visually dynamic way of representing function in DG microworlds and the influence of dynamic transformations of graphs allowed by FP in the students' perceptions of the property;
- the limitation, origin, usefulness of perceptions built by the students while discriminating, generalising and synthesising them;
- the associations made during their work and their progress;

- the influences of students' previous knowledge on these perceptions;
- the language students use while exploring the property in each microworld, in particular the use of terms learnt in school in trying to make sense of the property;
- the influence of the previous work in the other microworlds.

During the analysis, I look for the opportunities the students have for overcoming the limitations of their own built associations and/or knowledge-obstacles, which I will call critical moments. In my view, understanding what happened during these moments is crucial in analysing the students' paths of learning. I also believe that it is by overcoming associations and knowledge-obstacles that the biggest leaps occur in the progress of their perceptions.

The starting activities with DG Parallel provide an important source in the analysis of the knowledge-obstacles which derived from previous knowledge of functions. The differences in the students' perceptions before and after knowing that the strikers represent functions is a source for the analysis of these obstacles. The same source cannot be obtained in the starting activities with FP because of their similarities with school knowledge. The graphs and equations are used in both.

In the longitudinal analysis of the students' perceptions of each property, the blob diagram is used to present the development of these perceptions across the research environment and the final interview. This diagram is an adaptation of the one from Hoyles & Healy (1996), which presents information keeping the longitudinal approach. It gives to the reader a visualisation of the whole development of students' perceptions facilitating the analysis of the role of each perception in the whole study.

The blob diagrams will be presented, here, while constructing the diagram of the perceptions of constant function developed by Bernard & Charles, one of the pairs of students. This construction will be supported by appendix III, where all the reports will be presented in full. *The diagram shows each microworld (and pre-test) in one pentagon. The pentagons were displayed to allow two microworlds to be linked without passing through a third microworld and to keep the sequence of the microworlds.* In the case of Bernard & Charles, who followed the activities from DG to FP, the disposition as seen in diagram 7.1.

Diagram 7.1

Disposition of the microworlds in the blob diagrams

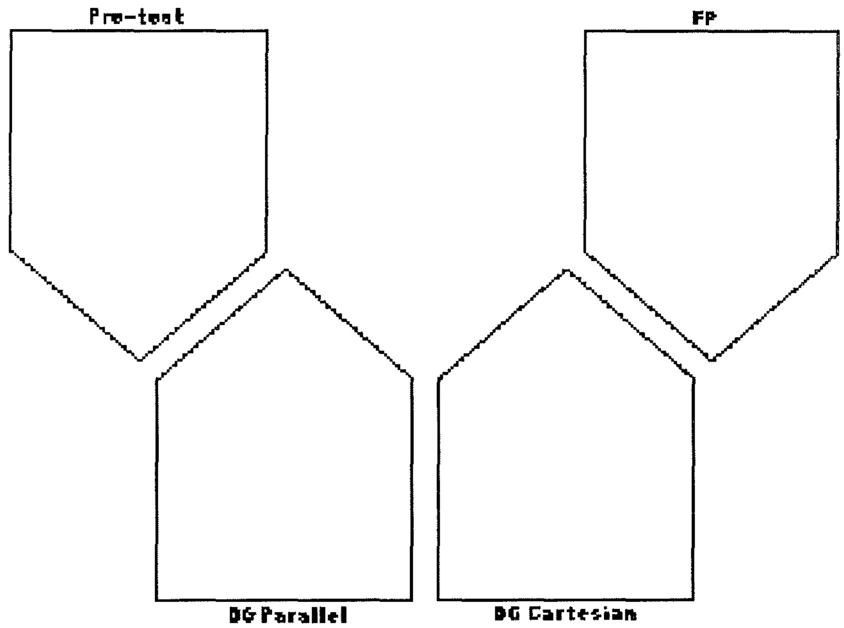
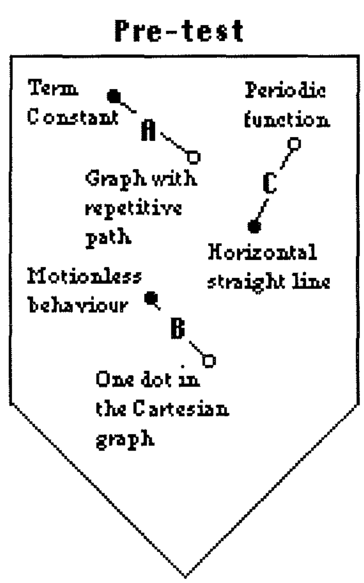


Diagram 7.2

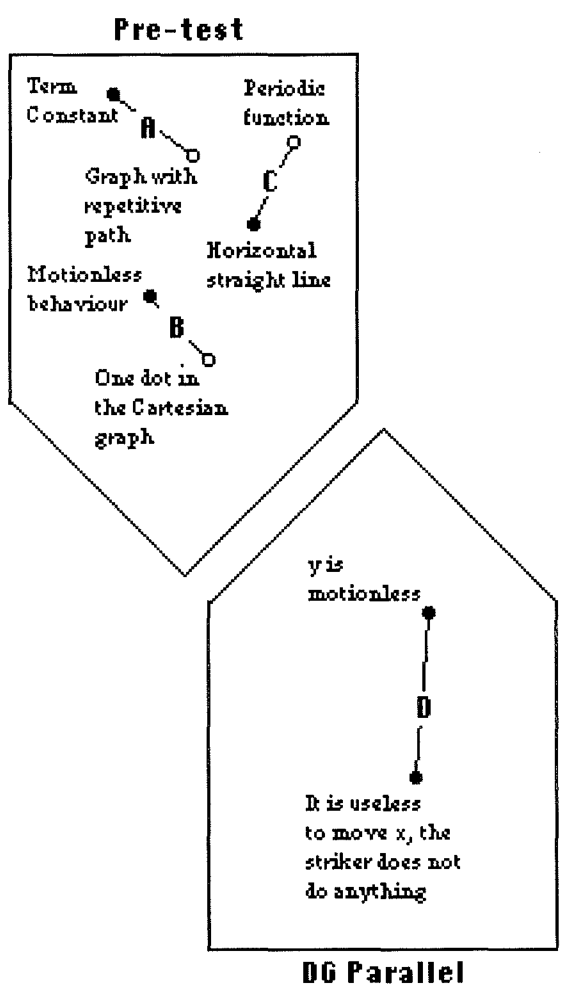
Pentagon of the pre-test



Each perception evidenced in the report (see appendix III) was represented by one blob. For example, Bernard & Charles defined the term constant function by a graph with repetitive path (see section AIII-1), thus, two blobs are put inside the pre-test pentagon, one for the term 'constant function' and one for 'graph with repetitive path'. Note that this perception has no correspondence to constant function from a mathematical viewpoint. *Cases like this led me to divide the blobs into two types: the* *full blobs and the blank blobs where the blank blobs indicated the views which had no apparent correspondence to a mathematical viewpoint of the property.* Thus, in the diagram of constant functions the perception 'graph with repetitive path' was represented with a blank blob. *As a topological diagram the position of each blob inside a pentagon has no meaning.* Both blobs were linked by a line labelled by A which represented this connection which is also an association between different ideas (see diagram 7.2). *The connections between different perceptions are shown by lines linking the blobs. Each link is denoted by one letter to enable me to refer to it in the text.* In the construction of the diagrams the evidence of each link was noted (see section AIII-5). In the same way, the other perceptions evidenced in the report (see section AIII-2) were represented in diagram 7.2. The diagram shows none of the perceptions presented by this pair has any correspondence with constant function from a mathematical viewpoint.

Diagram 7.3

Construction of the pentagon of DG Parallel



Then, the sessions with DG Parallel were analysed to build its representation in the blob diagram from the report in section AIII-2. As Bernard & Charles described both strikers corresponding to constant functions as being 'motionless strikers' a blob called 'y is motionless' was included in the pentagon of DG Parallel. Another blob labelled 'it is useless to move x, the striker does not do anything' was also represented, they were linked (see link D in diagram 7.3) because it represents an argument of the students while discussing their characterisation — 'motionless striker'. Note that the diagram clearly shows the separation between knowledge from the pre-test and those built in DG Parallel.

Diagram 7.4

Construction of the pentagon of DG Cartesian

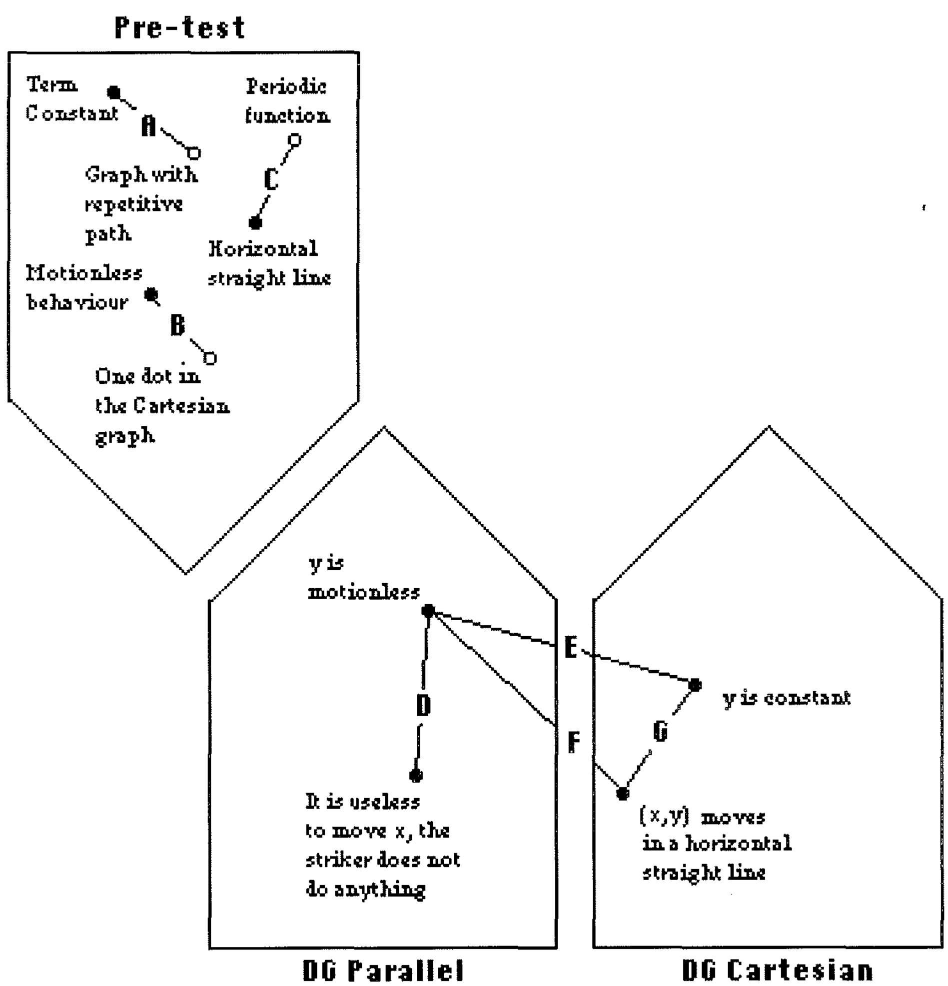
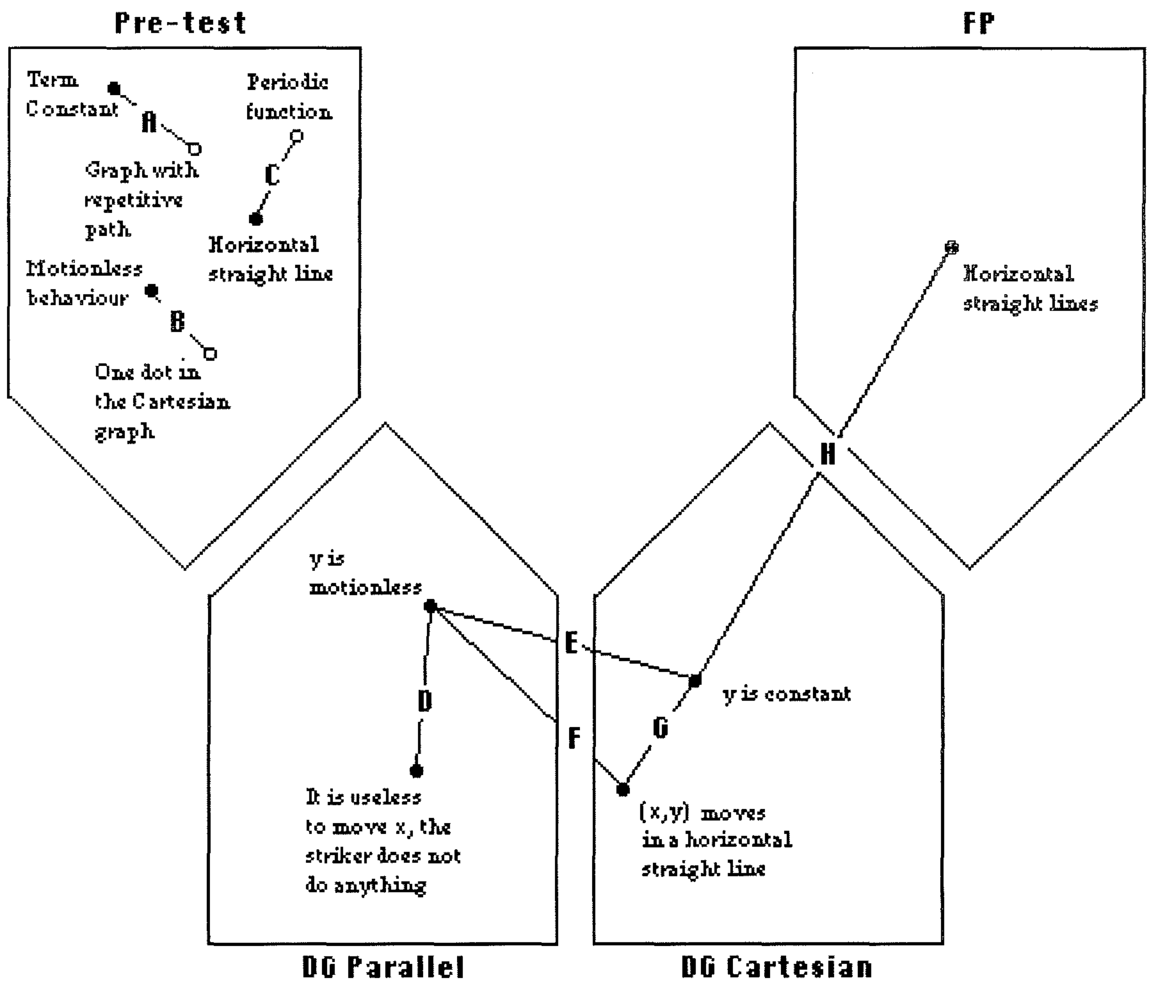


Diagram 7.4 shows that Bernard & Charles' perceptions of constant function in DG Parallel and DG Cartesian were linked but isolated from the ones exhibited in the pre-test. The idea of 'y is motionless' was brought to DG Cartesian when the students noticed that in 'the striker which (x,y) moves in a horizontal straight line' y was motionless, then constant. Two blobs were represented in the pentagon of DG

Cartesian: 'y is constant' and '(x,y) moves in a horizontal straight line'. The links E and F (see diagram 7.4) show the origins of both perceptions while link G was represented by the argument of Bernard & Charles (see section AIII-3).

Diagram 7.5

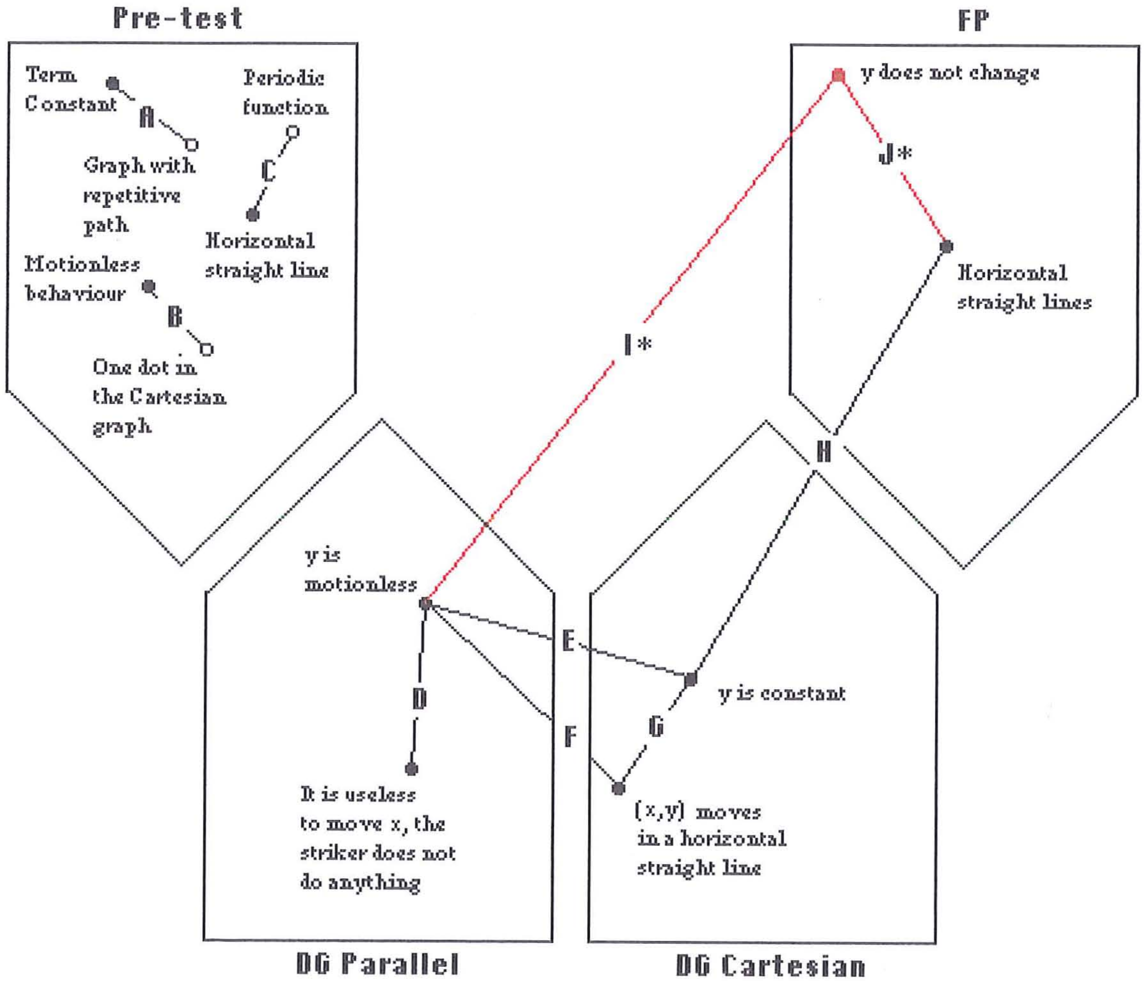
Construction of the pentagon of FP



The pentagon of FP presents only one blob called 'horizontal straight line' which was the way students characterised the graphs of constant functions. As they argued that the shape is due to the fact that 'y is constant' (see section AIII-4), link H was added. Thus, diagram 7.5 shows the continuity of Bernard & Charles' construction of the perceptions of constant function throughout the research environment, and also that they were isolated from their previous knowledge.

Diagram 7.6

Addition of the findings of the final interview



Then, an analysis based on the blob diagrams was written and results of the final interview were anticipated (see section AIII-5). Afterwards, the video-tapes of the final interviews were transcribed and analysed (see section AIII-6). In order to distinguish the perceptions and connections built in the final interview from those of the research environment, the red colour was introduced. *The lines and the blobs have two colours: black and red. The black ones will be used for perceptions and links built during the research environment while the red ones are for those built in the final interview.* In the final interview Bernard & Charles linked 'horizontal straight lines' with 'motionless strikers'. Thus, links J* and I* were represented in red. A new blob (red) was introduced in FP pentagon because of the students' explanation that the link is due to the fact that 'y does not change' in the graph. *The labels of the links are in alphabetical order but the motivated links are distinguished by an asterisk as a visual aid in the text.*

Finally, all the reports of the students for each of the properties (see the example in appendix III) were summarised in the longitudinal analysis which will be presented in chapter VII.

7.3 Cross-sectional analysis

The cross-sectional analysis is a comparison of the findings from the longitudinal analysis of the different pairs of students together with the analysis of the school approach to functions.

This analysis has a dual focus: the responses concerning each property of different pairs of students are summarised and categorised as synthesis, associations, knowledge-obstacles, and main features of each microworld that appeared to contribute to the students' progress. The blob diagrams grouped by properties are presented in appendix IV to help to compare the responses of the different pairs of students for each property.

In the first step, patterns of perceptions of the properties across the pairs of students are analysed. For each property, the analysis is divided according to the microworld in which the perceptions were developed. Thus, some variables on these patterns are considered such as sequence of microworlds used. In the second step, the important points observed in the longitudinal analysis were considered as starting points for building tables of patterns of students' interactions with the microworlds across the properties and pairs of students (see appendix V). From these tables, the findings were analysed and will be presented in chapter VIII.

V — The Pilot Study

The main study was designed in three phases. The first version of the empirical study was tried out with one pair of Brazilian youngsters. On the basis of the analysis of the data, I redesigned the experiment for the pilot study which was undertaken with three pairs of students: two pairs from a middle attainment level working in the different sequences of microworlds and one pair from a lower attainment level, who worked from FP to DG. The last pair of students were taken to determine the viability of the microworlds for students considered by their teacher to have great difficulties in learning mathematics.

Certain issues emerged from a longitudinal analysis of the work of the pairs of students which will be summarised here. These issues also directed the observation of the main study.

Classification of functions

The students' classifications of the functions usually matched with the families of functions. Nonetheless, two aspects influenced the students' recognition of these families: the sequence of microworlds and the microworld. For instance, in DG Parallel the students who began by working with DG used the perceptions derived from explorations of these microworlds such as 'y and x have proportional speed' and "y doesn't move while x does" as criteria in the classification while the other pair, who began by working with FP, used the family of functions as a criterion, because they had connected characteristics between the microworlds to sketch the graph corresponding to the behaviour of each striker. In the case of the classification session in FP, the shape of the graph and the equation were the strongest criteria for all the pairs of students.

Patterns in associations: pointwise perceptions and polarisation of knowledge

Some patterns in associations built by the students were identified. The students tended to associate the properties of variation with pointwise perceptions or rules involving polarisation of knowledge. Some examples of these associations are: "period of a sine function is the distance from zero (x=0) to the first root after two turning points"; increasing is the rule 'when x is positive, y is positive, when x is negative, y is negative'.

The pointwise perception and polarisation of knowledge seem to originate in the school approach to functions. I base this statement on the fact that all the pairs of students associated these properties with pointwise perceptions in more formal activities or when trying to link their perceptions derived from activities in the research environment with their previous mathematical knowledge. One student stated that variational properties belong to the strikers, and the pointwise or polarised properties belong to functions. Nonetheless, this statement needs further investigation.

Revisions of associations — counter-examples

The interactions of the dynamic microworlds, together with counter-examples of associations built by the students, allowed the students to realise these associations and to overcome the limits they imposed. A great difference between FP and DG microworlds regarding the revision of the associations was that in DG microworld the counter-examples of an association must be given while in FP the commands (translation, stretch, and reflection) allow the students to create their own examples and counter-examples. The students were able to overcome limitations of associations derived from pointwise perception by exploiting the dynamic transformations of graphs in FP while searching for function properties.

On the other hand, I must say that in many cases the associations remained. In some cases, counter-examples were missing. For example, one pair of students associated parabolas with 'a function which changes from increasing to decreasing or vice-versa once'. As the set of functions had not a counter-example for this association, for example an absolute value function, I cannot analyse the force of this association at critical moments. In other cases, I observed that the representation did not facilitate students' perceptions of some properties as well as revising associations. For example, while working in DG Parallel, none of the students revised the association between line symmetry and symmetric numbers.

DG Parallel as a 'new' representation

The activities of description and classification while searching for characteristics in the behaviour of the strikers associated to the fact that DG Parallel is a microworld where strong features (such as shape) are not present represented an interesting opportunity for the students to revise their previous perceptions of the function properties. Moreover, the exploration of this microworld gave them the opportunity to realise these associations and to overcome their limits.

DG microworlds served as a lens on students' perceptions

The interaction with DG microworlds facilitated the observations of obstacles students faced in developing their perceptions of the function properties, in particular, those resulting from the school approach to functions. As DG microworlds could be introduced without informing the students that they were working with functions, I observed how the students' perceptions of the properties as represented by the strikers changed as soon as they were told this. Their previous knowledge about function led them to consider motion as not belonging to the functions.

Obstacles derived from the school curriculum

Their school emphasis on algebraic representation during the introduction of the topic of functions seemed to have created an obstacle to the students' observations of other function properties. Equations seemed to be considered as the essence of functions. After guessing which equation represents the function of a striker, both pairs of students who began by working with DG stopped searching for function properties. This barrier was not observed with the other pair who worked in the inverse sequence. However, in the starting activity with FP this pair of students resisted analysing the function properties through graphs. Moreover, in FP I observed that all the students tried to characterise equations more than graphs.

The polarisation was very strong in the students' perceptions of the function properties. They often characterised a property of function as positive or negative. This tendency created obstacles when the students attempted to generalise the perceptions among different functions. For example, none of the pairs of students recognised any similarity between the strikers given by $y=0.25x^2$ and $y=0.25x^2-8$, even between their ranges, because their perceptions of range were categorised in positive and negative.

Two different barriers were derived from the approach the school gives to family of functions. One is close to the emphasis given by the school to equations. After recognising the family to which a function belongs, the students assumed that 'the family' was a complete characterisation of the function. They created a barrier against searching for more properties in the function, especially those properties that they had not studied at school in that family. Another kind of barrier arose from the students' over-generalisations of perceptions from a particular way of recognising a property within one family of functions. The students may have considered this over-generalisation correct because they studied some properties only in a special family of functions. For example, as the students studied minimum only in the family of functions which have turning point, they associated these two properties as being the same.

DG microworlds led students to develop variational perception

Some function properties are highlighted through the dynamic way DG microworlds represents functions. In particular, ideas related to variation of a function gain very different aspects. For example, the derivative gains the aspect of speed. In conclusion, it is the dynamic possibilities of DG microworlds which make its representation qualitatively different from the diagrams (see figure CIII-1.2) of **R** to **R** in paper-and-pencil representation.

Nonetheless, the way the students were enabled to perceive properties in exploring DG microworlds depended on the property. Monotonicity, derivative, constant function, turning points were easily identified in these microworlds. Symmetry was only observed by the students as symmetric numbers. Periodicity was only discriminated as a repetition of dots, such as: roots and turning points. The students did not recognise periodicity as being the repetition of the whole path of the striker.

The exploration of direct manipulation of x while observing the consequent behaviour of y in DG microworlds scaffolded a variational way of analysing some characteristics in graph in the pairs who began by working with DG. This way of analysing a graph was observed while they were working in FP. They used to analyse the growth of the functions in graph by moving their finger horizontally and seeing what would happen to y.

Transformations of graphs led the students to explore perceptions

The exploration of dynamic transformations of graphs in FP microworld allowed the students to check their own perceptions of the function properties. The activity of searching for properties to describe the functions together with the possible dynamic manipulations of graphs allowed all the students to realise by themselves their own associations, as well as to see different aspects of a property that they usually saw as being only one.

Different transformations of graphs emphasised different properties of the same function. This effect was so marked that one of the students thought that two graphs of the same function, which were obtained through different transformations, were two different functions despite overlapping. Therefore, I conjecture that each command structures a student's perception of a property in a different way. This perception also depends on the property that is being examined and on the function that the student is exploring. Nevertheless, this statement needs more investigation.

Interactions with FP modified students' preference for graphs

The interaction with FP scaffolded in the pair of students, who began by working with FP, a way to generate a function from a given function. I observed that on describing the strikers in DG Parallel, they were checking the accuracy of their descriptions by imagining translations, stretching or reflections in the behaviour of the strikers.

The interaction with FP redirected the students' attention from equations to graphs. The students who began by working with DG tried to connect the perceptions they themselves built in DG microworlds to equations while the pair who began by working with FP made the connection with graphs. Moreover, instead of plotting the graph, the pair who began by working with FP really sketched the graphs indicating characteristics of functions that they thought should be important, such as: monotonicity and slope for the linear functions and curvature for parabolas. Therefore, the findings suggest that the possibility of manipulating the graph in FP can change students' perceptions of functions, in particular the function properties. In addition, these connections represented evidence of spontaneous synthesis from the behaviour of the strikers to graphs and vice-versa.

VI — An Analysis of the Curriculum

This chapter will outline the structure of the Brazilian mathematics curricula and then analyse the way the students explored functions. The analysis of their work in the topic of function will have two focuses: how and for which families each of the function properties was studied; and which and how representations were used. Finally, I will discuss the expected over-generalisations and knowledge-obstacles in the students' perceptions of the properties.

1 The Brazilian mathematics curricula

Schooling in Brazil is divided into 8 grades of primary school, which all children are supposed to attend, and 3 to 4 grades of secondary schools. Under Brazilian law schooling is compulsory for children between the ages of 7 and 14, although as Brandão (1989: 743) argues, legislation for school reform does not solve the problems of education. Although the government tried to institute reforms to counteract the dualism of secondary schools, according to Werebe (1994) they are still divided into technical and academic schools. This study investigates the second kind of school. The academic school course takes 3 years, during each of which the students are evaluated to be up-graded or to repeat the same grade.

Education in Brazil does not follow a national curriculum, but the curriculum of each school is decided in stages. The national government decides the minimum number of hours for a minimum core of subjects. Each state determines for its own schools the other subjects as well as the topics that the schools should follow. The private schools in general follow the topics determined by the state adding some other subjects and topics depending on their aims. Although Brazilian schooling has not a national curriculum, the use of the textbooks in some ways gives uniformity to the approach to some subjects such as mathematics.

In Pernambuco, the Brazilian state where this study was undertaken, the educational committee determines the general aims of mathematics, the topics as well as the minimum content for mathematics in each grade (Secretaria de Educação de Pernambuco, 1986). Despite the claim that the aim of teaching mathematics is to enable students to use it in everyday life and that the students' intuitive knowledge must be taken into consideration, this is not the reality of the mathematics classes.

The teacher of the selected students, for example, follows a very formal mathematics course. He admits that despite considering the contextual and intuitive to be the best approach to teaching mathematics, it requires more time than the formal one. Therefore, he has to follow the formal mathematics course in order to cover all the minimum content.

The teacher says that he uses Iezzi et al (1990) with students as a textbook for the basic curriculum material. He describes his mathematics course as being lectures with a form of seminar given by himself following the sequence of the textbook which students then read and resolve the problems from it. He says he rarely prepares any kind of other activities. Therefore, this analysis is based on this textbook and some students' notebooks. In the following sections the quotations with no specified source are from the textbook.

2 Previous work

Comparing the grades in English and Brazilian schooling, the first grade of the Brazilian primary school corresponds to the third year in English schooling. In fact, in Brazilian schooling there are two pre-primary grades which are not compulsory. The school chosen for the investigation only has from fifth grade of primary schooling to third grade of secondary schooling. For these grades the mathematics curriculum includes the following topics:

Primary school:

Fifth grade: Natural numbers, positive rational numbers (decimal and fractionate representation), measures (length, area, volume, height, mass) and geometry (terminology and classification);

Sixth grade: integers and rational numbers, proportionality and geometry (angles, construction of triangles);

Seventh grade: real rational numbers, algebra (systematic description of geometry) and measures (area and volume);

Eighth grade: power and roots, equations (first and second degree polynomials), linear and quadratic functions, geometry (similarities, Pythagoras' theorem, metric relations in a circle and regular polygons), measure (cylinder, cone, sphere) and trigonometry (right-angled triangle).

Secondary school:

First grade: set theory and theory of functions (first and second degree polynomials, absolute value, exponential and logarithmic, composition of and trigonometric functions);

Second grade: matrix, linear systems, probability, Newton's binomial theorem and spatial geometry (prisms, pyramid, cylinder, cone, sphere and polyhedrons (Euler's formula and regular polyhedrons)
Third grade: analytic geometry, complex numbers, polynomials, equations, revision.

According to this curriculum students are introduced to the concept of function in the eighth grade of primary schooling, and study the notion again in the first grade of secondary schooling. In eighth grade of primary schooling, students work with first and second degree polynomials and then they are introduced to the notion of functions given by first and second degree polynomials. At this stage, according to their teacher, students have studied how to plot graphs from equations. Therefore, the first approach is functions given by equations.

In the first grade of secondary schooling, students study functions during the whole year. First they are introduced to the notion of sets. Before being introduced to the topic of function, students study binary relations and the Cartesian system with emphasis on working on algebraic relations.

3 The introduction of the topic of function

The selected students were introduced to function in two ways: as an 'intuitive notion' and as a 'mathematical notion'. As intuitive notion of function, the textbook presents many examples of contextual relations between two variables which compose a function, such as: the population of a country is a function of the historical time, the area of a circle is a function of its radius, the price we pay for the petrol we buy is a function of the number of gallons we put in the car, ... Then it introduces function as the relation between two quantities x and y such that

> "for each value given to x there is, correspondingly, only one value associated to y". (p.38)

In this introduction, the textbook uses tables to give examples of relations which are functions and relations which cannot be functions. The proposed exercises explore tables to interpret derivative and monotonicity. The only representation used is tables.

After the intuitive notion of function, the authors introduce ordered pairs (showing in figure 3.1) which they call mathematical notion of function, followed by the explanation:

"The relations R_1 and R_2 present a particularity, for all elements of A, they associate only one element of B, which does not happen with R_3. Relations such as R_1 and R_2 are called functions or applications". (p.42)

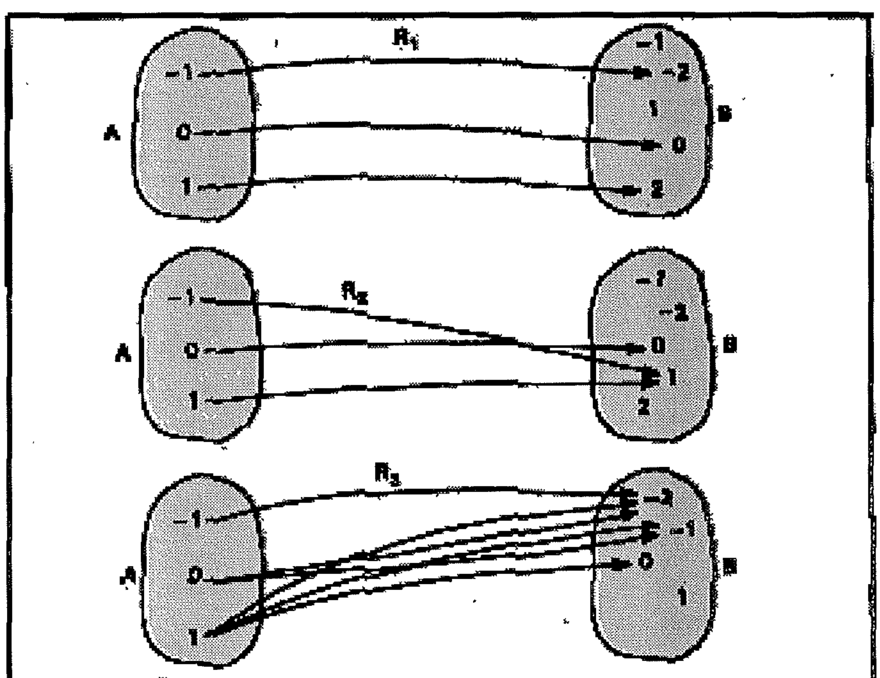
Therefore, the students were introduced to function as being a special case of binary relation.

Their teacher said that he emphasises functions as being "two sets and a rule associating the two sets". According to him, he never emphasises the use of definitions. He then said that after functions are introduced as a particular case of binary association, the students work with families of functions toward the construction of the graphs to use in solutions of inequalities.

Figure 3.1

Introduction of the mathematical notion of function in the textbook (p.42)

$R_1 = \{(x, y) \in A \times B \mid y = 2x\}$
$R_2 = \{(x, y) \in A \times B \mid y = x^2\}$
$R_3 = \{(x, y) \in A \times B \mid x > y\}$



After the above-mentioned introduction, the textbook introduces algebraic notation of functions followed by the introduction of graphic notation, which is divided into 'intuitive graphs', 'construction of graph' and 'recognition of graphs of functions'. In 'intuitive graphs', students are asked to plot graphs from verbal and contextual description while in 'construction of graphs' they are asked to plot graphs from equations passing through tables, plotting the points and joining them with the shape corresponding to the family of the equation.

After this general introduction, functions are studied compartmentalised into families. The families of functions are determined by the algebraic expression. This is very clear by the way each family is denominated. For example, the family of linear and quadratic functions are called the first degree polynomials and the second degree polynomials, respectively. The textbook also denominates by linear only the functions given by y=ax, emphasising linearity in an algebraic sense: a function is linear if and only if $f(Dx)=Df(x)$ for any real number D and $f(x_1+x_2)=f(x_1)+f(x_2)$ for all x_1 and x_2 belonging to the domain of f, instead of stating that linearity means a straight line.

According to their teacher the selected class studied: first and second degree polynomials, absolute value functions and trigonometric functions. The family of exponential and logarithmic functions were studied only in algebraic properties. So the table below shows the kind of exploration the students made in each family of functions they had studied.

Table 3.1

Properties explored in each family of functions

	General introduction	Linear function	Quadratic function	Absolute value	Trigonometric function
Turning point			DHE	H	H
Constant function		D			
Monotonicity	DGE	DGT			C
Derivative		DGE			
Second derivative		D	E		
Range	DG	D	DHE	DH	DGE
Line symmetry	DHE		GH		DG
Periodicity					DHE

(D) Discussed algebraically; (G) Discussed graphically; (T) Highlighted with table; (H) Highlighted in graphs; (E) Explored in problem-solving and (C) Discussed only in the classes

As table 3.1 shows different properties were emphasised in different families of functions. The following section will report in detail the emphasis given to each of the chosen properties in the different families these students worked with.

4 The properties and families of functions

As table 3.1 shows, *the concept of turning point* was introduced to the students in the family of quadratic functions. The first meaning of turning point presented in the textbook is linked to extreme values through examples. After presenting the graphs of parabolas with the turning point highlighted, the textbook continues:

> "Among the points of the parabola $y=-x^2+2x$, the one with maximum ordinate is (1,1), it is denominated by turning point of the parabola". (p.85)

Table 3.1 also shows that in the other families of functions the turning point is presented only as a highlighted dot in graphs. This approach can lead the student to perceive turning point as being a special point in Cartesian graphs.

Still in the chapter on quadratic functions, the idea of turning point and the sign of the coefficient 'a' in the formula $f(x)=ax^2+bx+c$ are used to decide whether a turning point determines a maximum or a minimum. At this point, the textbook develops an algebraic formula to calculate the coordinates of the turning point.

The turning point is also used as a way to recognise line symmetry in parabolas and sine graphs. The textbook says:

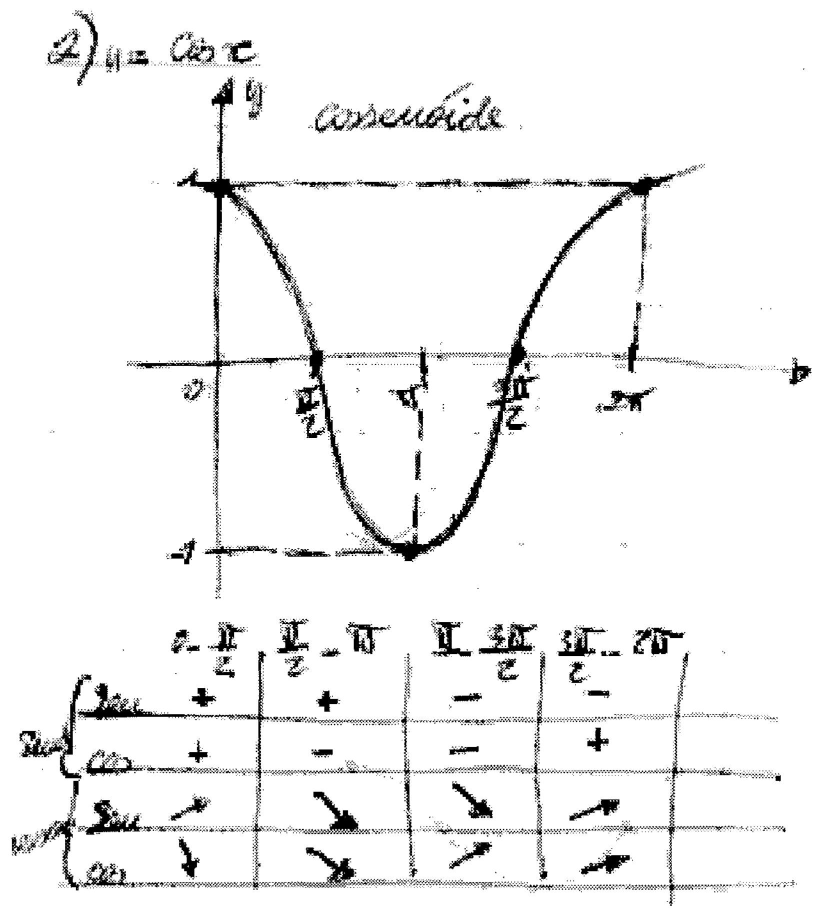
> "a parabola presents a line of symmetry, that is a straight line parallel to the y-axis passing through the turning point". (p.89)

The only time that turning point was presented to the students as being 'the point where the graph changes direction' was while exploring 'the domains where a function is positive and negative' for sine functions in the students' notebooks. They made a table and a graph highlighting the special points: roots and turning points. In the table, they indicated with arrows the direction of the graph for each interval between special points (see figure 4.1).

The difference in terminology in the English and Brazilian curricula should be clarified: the English term 'turning point' suggests 'the point where something turns'; in the Brazilian curriculum the word used for turning point is 'vertex' and this word is used in two different topics of mathematics (geometry and function) with different meanings.

Figure 4.1

Exploration of monotonicity and points in the classes (student's notebook)



The concept of constant function was introduced to the students as a particular case of the first degree polynomial. The textbook writes:

> "When a=0, the function f(x)=ax+b is such that f(x)=b for all real x. In this case f is said to be constant function". (p.67)

After that, it presents the graph of y=2 with the point (0,2) highlighted as an example of a constant function. This fact can lead the students to perceive the graph as the point at (0,2). Constant functions are not explored further in the textbook, even in the exercises proposed.

Note also that the introduction of constant function has no reference to a function with derivative zero. This is another example of compartmentalisation of knowledge in school mathematics.

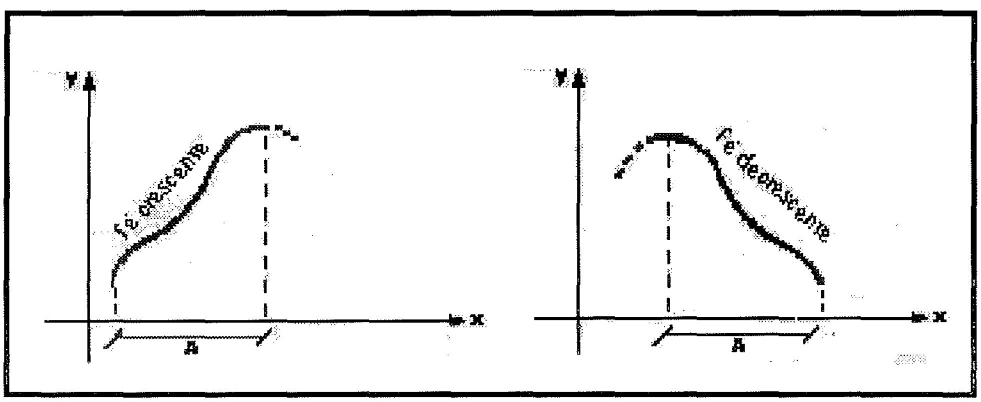
The students are introduced to the notion of *monotonicity* from the general introduction of function (see table 3.1). After presenting all the representations of functions they will work with, the textbook introduces some function properties such as even and odd functions, monotonicity and line symmetry. The notion of monotonicity is presented after working with functions given by formula. The textbook first defines the increasing function by:

> "If for any elements x_1 and x_2 of a set A, such as $x_1 < x_2$, we have $f(x_1) < f(x_2)$ then f is increasing in A; if for $x_1 < x_2$, we have $f(x_1) > f(x_2)$, then f is decreasing in A". (p.58)

In the graphic example of monotonicity (see figure 4.2), the textbook does not limit the notion of increasing to functions which increase in the whole domain, nor to linear functions. The notion is defined for functions with curvature and in part of the domain. However, the textbook presents only a pictorial view of how an increasing graph will look by highlighting the increasing part of the graph. In the general introduction, the textbook highlights the increasing and decreasing parts of the functions on different kinds of graphs. In the set questions, it explores the interpretation of these properties in graphs and equations. It also tries in the questions to distinguish the domain where a function is positive from the domain where it is increasing.

<u>Figure 4.2</u>

<u>Graphic introduction of the idea of monotonicity (p.58)</u>



(f is increasing; f is decreasing)

After the general introduction, the textbook limits the exploration of monotonicity only for linear functions, which is increasing or decreasing in the whole domain. In contrast, the students' teacher discussed the property for trigonometric functions in classes. He introduced another notation to indicate increasing or decreasing (see figure 4.1).

In the family of linear functions, a table followed by a graph is used to introduce monotonicity (see figure 4.3). At this point, the notion is presented in more informal language:

"Given the first degree polynomial y=-x+1, we can observe that as the values of x increase, the values of y decrease correspondingly; that is why we say that the function is decreasing". (p.72)

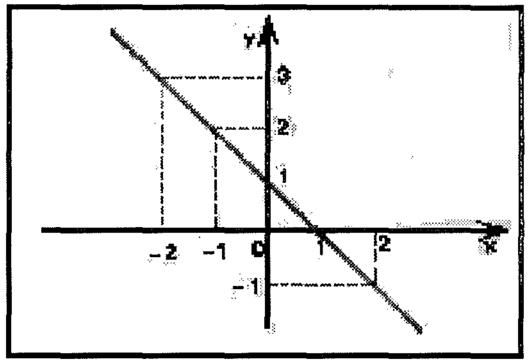
Figure 4.3

Graphic and tabular introduction of the idea of decreasing (p.73)

aumentando x

x	-2	-1	0	1	2
y	3	2	1	0	-1

diminui y

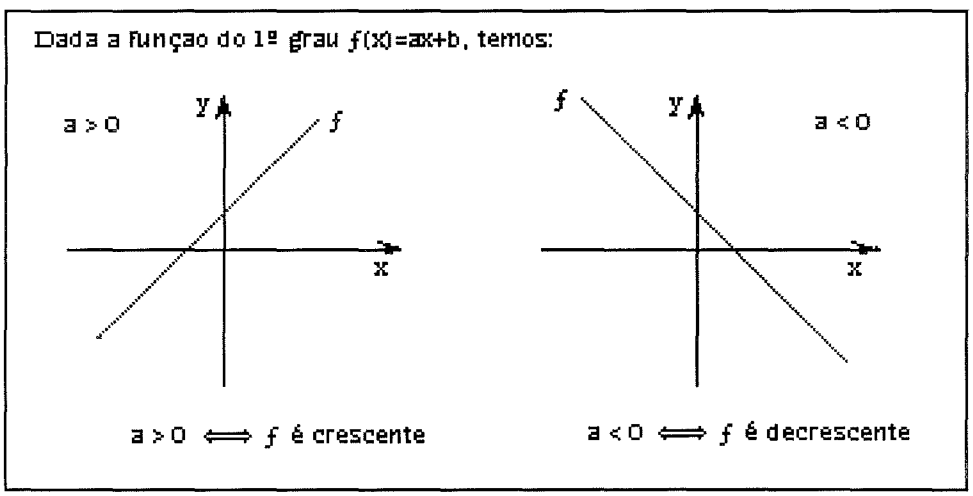


(increasing x; y decreases)

The textbook uses the definition of monotonicity to prove the connection between the sign of the coefficient 'a' in the formula y=ax+b and monotonicity. Then the authors summarise the connections as

Figure 4.4

A summary of the connections of monotonicity in all representations (p.73)

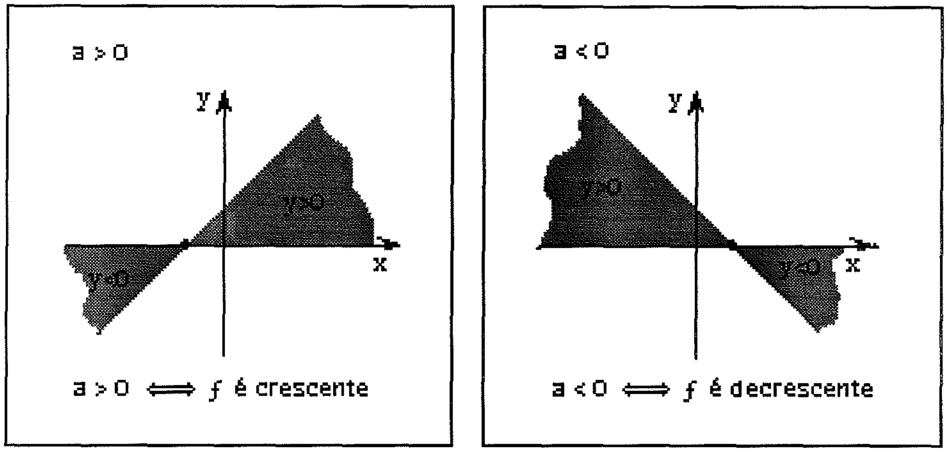


(Given a first degree polynomial f(x)=ax+b, we have: a>0 <=> f is increasing; a<0 <=> f is decreasing)

When investigating 'the domains where a function is positive or negative', the textbook presents figure 4.5 which seems to be a source for associations between increasing function and 'y is positive to the right side and negative to the left' and vice-versa for decreasing.

Figure 4.5

Scheme of sign of the values of f for linear functions (p.74)



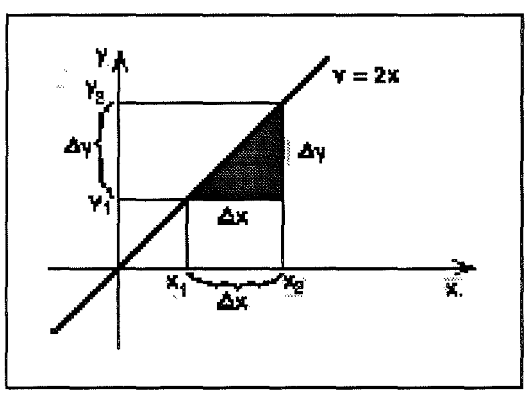
(a > 0 <=> f is increasing; a < 0 <=> f is decreasing)

Table 3.1 demonstrates that the notion of *derivative* was introduced to the students only in the family of linear functions by introducing the formula of 'rate of average change' which coincides in the case of linear functions with derivative. The textbook says:

> "If f is a numeric function and x_1 and x_2 are two elements of the domain such that $x_1 < x_2$, we call $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ rate of average change between x_1 and x_2 of the function f in relation to x". (p.69)

Figure 4.6

Graphic presentation of rate of average change in the textbook (p.70)



After introducing the formula to calculate rate of average change, the textbook presents calculations from linear equation proving that rate of average change is equal to the linear coefficient. It also introduces graphical examples of the meaning of rate of change (see figure 4.6). Many of the set questions aim to make the students link the linear coefficient to rate of change. However, there is no attempt to make the students connect the inclination of the graph to the linear coefficient. This attempt was found in the notebooks copied from lectures. There the students tried to link the coefficient of y=x to the angle formed by the x-axis and the graph, in particular to the measurement of the angle. In the textbook, the proposed exercises do not explore the graphic representation, nor do they explore the tabular representation in the notion of derivative.

The interesting fact is that the notion of monotonicity is discussed after the notion of rate of change without linking the two notions. They are completely compartmentalised.

As soon as the textbook introduces derivative the students are asked to observe that: if the rate of change of a function is constant, then the graph of this function is a straight line. This is the beginning of the idea of second derivative. In other words, it says that 'if the derivative is constant, the graph is a straight line, otherwise it is a curve', but this notion seems to be stated without being related to the students' previous knowledge.

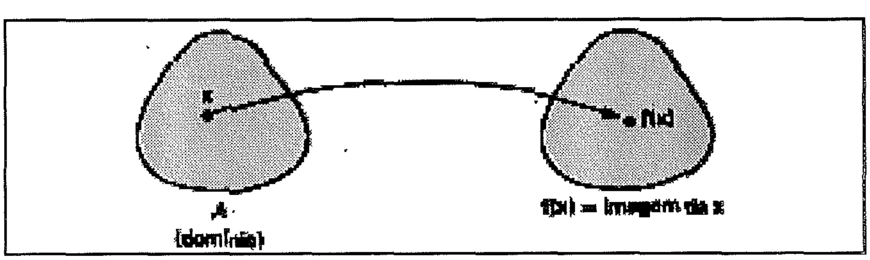
The textbook does not emphasise *curvature* for quadratic functions (second derivative). This concept is only marginally explored when students are asked to trace graphs of three quadratic functions with different curvatures. Therefore, the notion of *second derivative* is explored in two ways only: when it is zero the textbook links it to the form of the graph as I explained above and by talking about curve in the other families of functions. Nonetheless, in the other families there is no discussion about curve and 'variable rate of change'. Even 'rate of change' is not explored except in the chapter on linear functions.

As soon as the students have been introduced to 'the mathematical notion' of function (see section 3), the textbook discusses the notion of domain and *range* of a function. It defines domain and range in the following way:

> "The set A of the values of x is called domain of the function. The value of y corresponding to a value of x is called image of x by the function, or the value of the function in x, and it is represented by f(x)". (p.42)

<u>Figure 4.7</u>

<u>Diagrammatic presentation of domain and range in the textbook (p.42)</u>

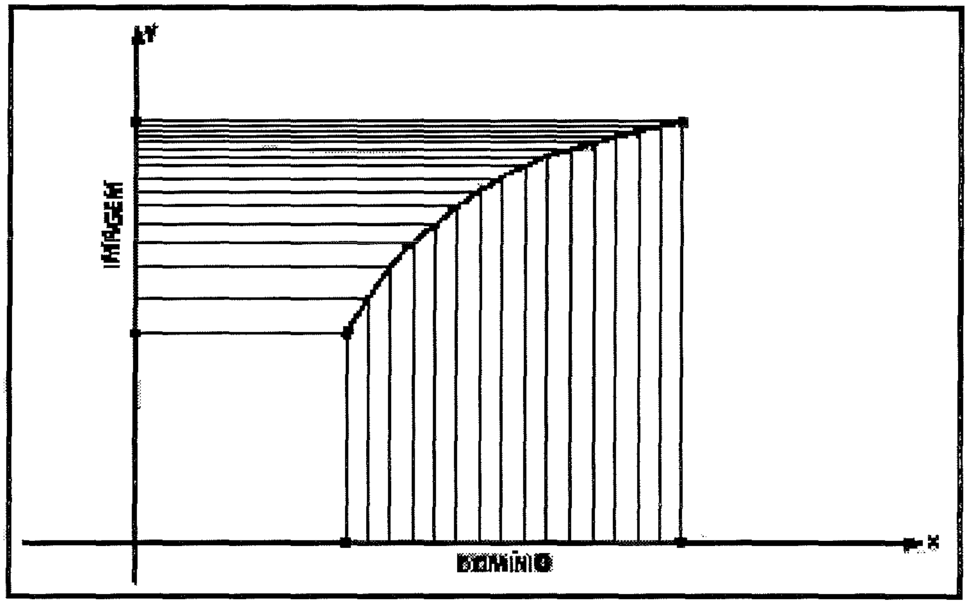


"The values of the images of $f(x)$ compose a set 'Im' called range of A through the function.
The range is always a subset of B". (p.43)

Using the graph in figure 4.8, the textbook introduces a method of determining the range of function in a graph.

Figure 4.8

Graphic definition of range and domain in the general introduction (p.53)



(range; domain)

After this brief introduction, the textbook explores the range in each family of functions. For first degree polynomials, it is stated that

"the domain of a first degree polynomial is **R** and the range is also **R**". (p.67)

This section also includes a brief classification of three kinds of linear function: affine, 'linear' and constant. In the case of constant functions the range is identified as being the set {b} when the function is given by $f(x)=b$ and shown in graph. In the proposed questions the idea of range is not explored further. This seems to be a very brief reference to the idea for this family of functions. In fact, the only families for which this notion is further explored are quadratic and trigonometric function.

In the families of quadratic and trigonometric functions, range is explored in the set questions when students are asked to calculate the minimum and range of the functions using equations. For the other families of functions, range is only defined. In conclusion, the idea is stressed in functions with turning points.

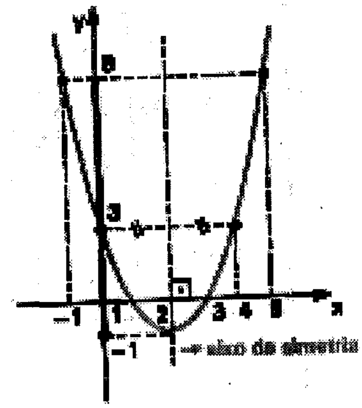
For parabolas, range is introduced linked to turning point and extreme values. With an example using equation (to calculate the turning point), the textbook says:

"the domain of the function is D=**R** and the range is Im={yERly≥-1} (see the graph)". (p.90)

Figure 4.9

Exploration of range in graphical and tabular representations (p.90)

x	y
-1	8
0	3
1	0
2 (Vértice)	-1
3	0
4	3
5	8



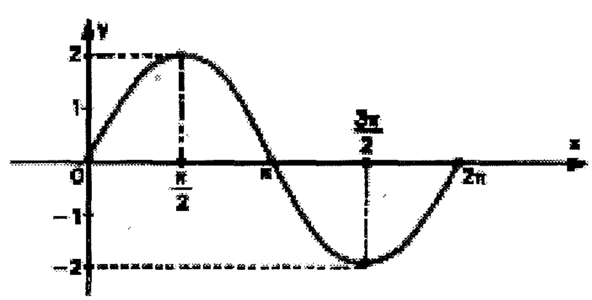
(Turning point; line of symmetry)

In the case of sine functions, range is discussed after presenting sine function through table, graph and equation. The range of $f(x)=\sin(x)$ is discussed as being the real numbers between -1 and 1. It is interesting that the sine functions are the only type of functions for which the textbook details the analyses of range. It presents a section of sine function with translated and stretched sines by equation, table and graphs where it discusses range among other properties (see figure 4.10). The idea of range is also explored in set questions.

Figure 4.10

Graphic and tabular explorations of range in the textbook (p.202)

$y=2\sin(x)$



Resolução:

x	y
0	0
$\frac{\pi}{2}$	2
π	0
$\frac{3\pi}{2}$	-2
2π	0

D = **R** Im = [-2, 2] p = 2π

The same work is made with the functions of figure 4.11. The table here tries to relate the influences of coefficient on domain, range and period.

Figure 4.11

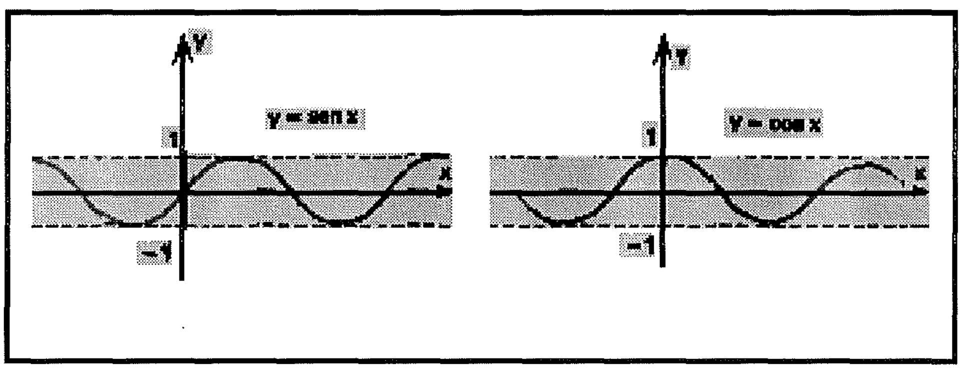
Table relating coefficients, domain, period and range of sine functions (p.209)

função	domínio	imagem	período
a) $y = 1 + \text{sen } x$	$\mathbb{R}$	$[0, 2]$	2π
b) $y = 2 \text{ sen } x$	$\mathbb{R}$	$[-2, 2]$	2π
c) $y = \text{sen } 2x$	$\mathbb{R}$	$[-1, 1]$	π
d) $y = \text{sen } \left(x - \frac{\pi}{4}\right)$	$\mathbb{R}$	$[-1, 1]$	2π

(function; domain; range; period)

Figure 4.12

Graphic presentation of a limited range (p.206)



Also for sine functions the textbook defines a *bounded function* graphical (see figure 4.12) and verbally. It argues that 'a function is bounded if there is a positive number M such that $|f(x)|<M$'. Upper or lower bounded functions are not explored. The authors present a parabola and a linear function as not being bounded.

In the topic of function, the students were introduced to *the concept of line symmetry* during the general introduction by using line symmetry in the y-axis on graph to introduce the idea of even function algebraically. That is, the authors present even function saying that it is a 'function that $f(x)=f(-x)$ for all x, so f has a symmetric graph', then they show a graph as visual feedback. Afterwards the textbook discusses line symmetry in the x-axis as well as symmetry in relation to the point (0,0). This leads to the definition of symmetric properties in algebraic representation with a feedback on graphic representation.

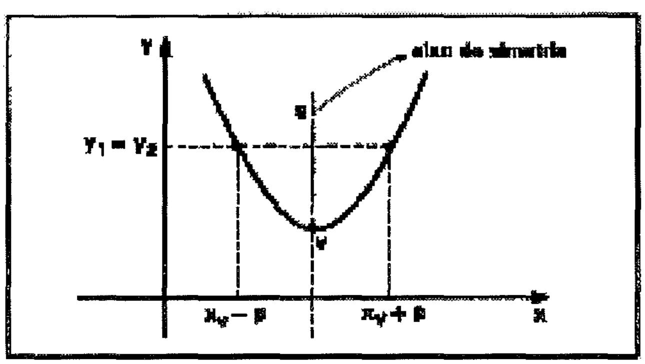
Note also that all the kinds of symmetry discussed above are in some way linked with positive and negative numbers. It is interesting that the notion of line symmetry arises discussed again in the curriculum when it deals with the line of symmetry in graphs of parabolas. At this point, line of symmetry is drawn when different from the y-axis, but the pointwise correspondence is not mentioned. Note that in this case, the symmetric numbers do not work. This is also the first time that line of symmetry is traced in graphs. On exploring inverse function, line symmetry on y=x is also discussed by its pictorial perception on graphs.

Another point is that the line of symmetry appears in the section dealing with the calculation of abscissa of turning point. After concluding the formula for this calculation, the textbook remarks:

> "It is important to know that the parabolas present a **line of symmetry**, which is a straight line(s) parallel to the y-axis passing through the turning point of the parabola." (p.89)

Figure 4.13

Graphic presentation of line of symmetry different from the y-axis (p.89)



(line of symmetry)

Note that this can originate an association between line of symmetry and turning point. This also indicates a compartmentalisation in the students' perceptions of line symmetry in graphs and line symmetry in a pointwise way.

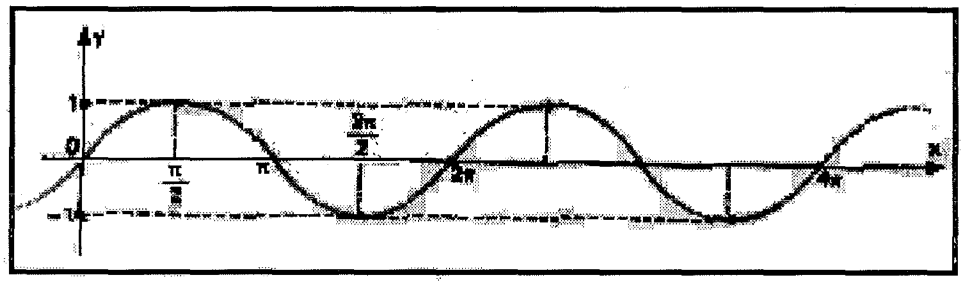
The idea of line symmetry in the y-axis and point symmetry is discussed again in the family of trigonometric functions while defining even and odd functions. Here, the textbook gives a formal definition and graphic examples of these notions.

The students were introduced to the idea of *periodicity* when studying trigonometric functions. After introducing trigonometry as relations of sides in a right-angled triangle and studying the representation in the trigonometric cycle, their textbook presents sine functions with their graphs. Arguing that after 2π the function starts to repeat its values showing 'special points' highlighted in the graph, the textbook presents the notion of periodic function is presented as being

> "the function that behaves in a similar way to the sine, i.e., repeats its variation". (p.201)

Figure 4.14

The graph of sin(x) (p.201)



Soon after this introduction a formal definition is presented and this is done more in relation to algebraic representation. The textbook states:

> "a function f:A->B is periodic if there exists a positive number p such that f(x+p)=f(x) for all x in A". (p.201)

Some questions arise: do the students connect this formal definition to the graph of a periodic function? Or will they maintain a pictorial perception of periodicity?

The above definition is followed by the definition of the period:

> "The smallest positive value p is said to be the period of the function f. Intuitively, period is the length of the smallest interval in which the function completes a cycle". (p.201)

Although the corresponding idea is discussed in the text, it is not shown in graph.

Figure 4.10 shows one from the four examples of sine functions in which the textbook discusses period and range in equation, table and graph. Once more in all examples the period is calculated from a special point: x-intercept or turning points. The periodicity of cosine and tangent functions is also explored in the same way as sine, with little emphasis. The exercises are designed to calculate the period from the equation after the textbook relates coefficients and period.

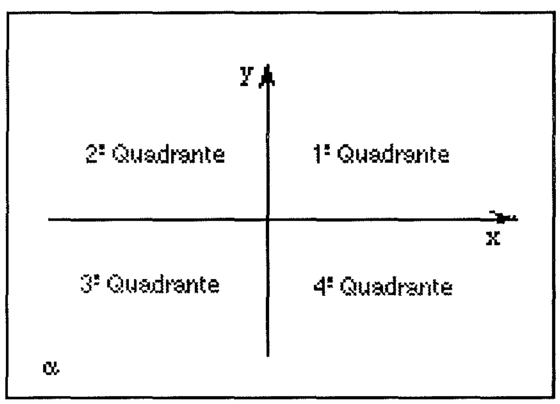
5 The role of each representation

It is the algebraic representation which is given most emphasis in the school mathematics. As the teacher argued in the interview: "we don't emphasise the construction of the graph. When we are working with the people [students], we want to use function as a tool for our algebraic work". This emphasis on the algebraic representation is also observed by the division into families of functions. Also on dealing with properties of functions, the algebraic representation is more often used as an action representation. For example, the textbook does almost all the work on turning points in algebraic representation in order to build the Cartesian graph. This last representation is used more as feedback than as an action representation.

The students were introduced to the Cartesian system before functions. The textbook explores the Cartesian representation dividing it in four quadrants, presented by figure 5.1. In this introduction, the students studied how to plot points.

Figure 5.1

Graphic division of the Cartesian system in quadrants (p.29)



(1st quadrant; 2nd quadrant; 3rd quadrant; 4th quadrant)

In the general introduction to functions, the students studied how to plot graphs of functions from a verbal description through the use of tables. Note that the students constructed graphs by plotting them, they were requested to sketch graphs only when working with inequalities. According to their teacher, the families of functions were studied based on equations leading to the construction of graphs, in order to use graphs to work with inequalities. In addition, by analysing the way the textbook explores each property of function, I observed that graphs always follow the discussion of a concept as a visual feedback. The textbook does not discuss a concept in graphic representation.

The work in constructing graphs was reported by the teacher to be very brief. He argued that the students had already studied this in the tenth grade while working with linear and quadratic equations. He tries to make the students understand how to trace a graph from critical points in each family of functions. That is, for linear functions, the students learnt to find x-intercept and y-intercept, to plot the dots and to link them with a straight line. In the case of parabolas, the critical points are roots and turning points. This is also the emphasis of their textbook. It is not clear that the students understand why each graph had the shape they have drawn linking the points.

The work on graphs is more detailed for trigonometric functions, where translations and stretches relating graphs and equations are explored. Moreover, when constructing the graph for the first time, the textbook uses many points to show the students which shape the points will form. After that, special points are again introduced as a way to sketch graphs. Graphs are also treated as the final representation.

Despite more stress being on algebraic representation, for some properties, line symmetry for example, Cartesian representation has a different role. In algebraic representation, the textbook discusses a pointwise sense for line symmetry in the axes, while in the Cartesian representation this line symmetry is extended to line of symmetry different from the y-axis without discussing the pointwise sense of this generalisation. It is only highlighted in the graph.

According to their teacher, algebraic and graphic representations are the only representations he explored in classes. Looking at the notebooks, I observed that in fact tables were used as a passage from equations to graphs. In other words, the teacher used tables only to take notes of points from calculations with equation in order to plot later in a Cartesian system. Only once he used a table taking notes from a graph which was the one referring to the idea of monotonicity in figure 4.1. In the general introduction to function, while working with what the authors call 'intuitive notion of function', the only representation used is the table. Afterwards the table is used as a bridge between equation and graph. In other words, giving the Dirichilet-Bourbaki definition of function, table is used to organise and calculate the coordinates and to trace a graph. After that, the table is used as an action representation only twice. First, on dealing with the notion of monotonicity, the students use tables to recognise whether a linear function is increasing or decreasing (see figure 4.3). Second, turning points are indicated in tables.

What is really interesting is the change in the emphasis on table as a representation of function from 'the intuitive' to the 'mathematical' notion of function. In the intuitive the table is used as a source of the analysis of properties of functions as well as to understand the function. In the 'mathematical' notion the table assumes the role mentioned in the preceding paragraph.

6 Over-generalisations and obstacles

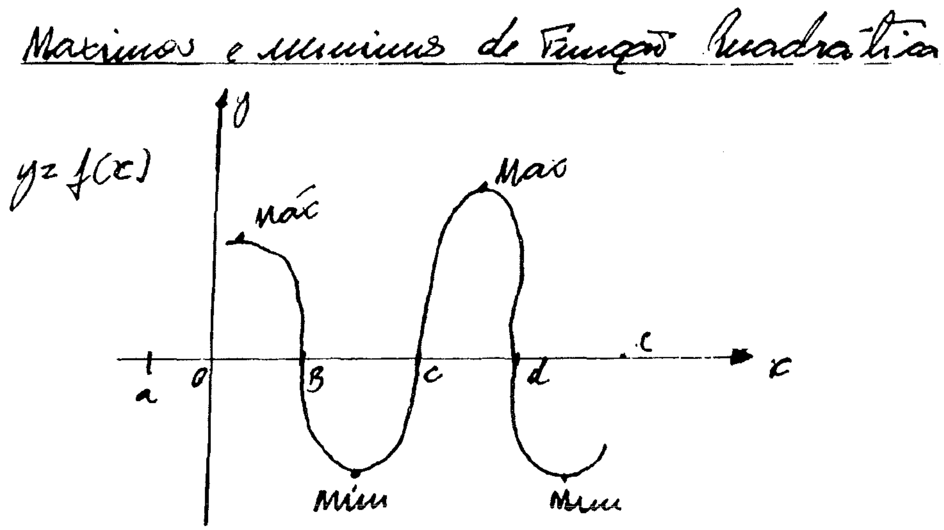
On introducing the term constant function, there is no discussion of 'what' is constant. The students can interpret that: the point (0,2) is the constant, $f(x)$ is the constant, and x is the constant because it does not appear in the equation. Moreover, the term "constant" is also used to characterise the derivative of linear functions.

As linear function was also studied in the primary school, this family of functions is over-emphasised in the secondary school. Therefore, the students can perceive monotonicity restricted to linear functions: as being some rules involving positive and negative suggested by figure 4.5; as being the sign of linear coefficient linked with the direction of a graph; and compartmentalised from the idea of derivative. The restriction can generate barriers when the students should generalise monotonicity to other families of functions.

The fact that turning points are only discussed for quadratic function can induce a strong link between turning points and parabola in the students' perceptions. On analysing one students' notebook, I found a passage where he denominates a graph of a sine function as being quadratic function.

Figure 6.1

Student's notebook treating a sine as a quadratic function



(Maximum and minimum of a quadratic function)

As rate of change is only explored for linear functions, the students can construct the link between rate of change and angle between straight line and the x-axis. This link can limit the idea of rate of change to the linear functions.

The use of the same ‘a’ to denote the angular coefficient as well as the linear coefficient can lead the students to over-generalise increasing as being positive curvature for parabolas. The fact that they do not explore monotonicity in the family of quadratic functions can contribute to this over-generalisation.

Their mathematics curriculum presented a clear preference for exploring polarised notions while neglecting the order of the notions. This is evident in the emphasis it put on monotonicity for linear functions, positive or negative curvature for parabolas together with the limited exploration of derivative and curvature. Moreover, as their teacher argued, the students were always directed to study inequalities, in which they usually had to verify whether a function was positive or negative. This can be another source of this tendency for polarisation of the properties when dealing with functions.

Emphasising range for functions with extreme values can generate a perception of range restricted to bounded functions. Moreover, the stronger emphasis given to range in sine and cosine functions can lead the students to consider only bounded range.

By the compartmentalisation in the discussion of line symmetry, I expected a gap in students' perception of line symmetry in graphic and algebraic representation. As mentioned above, line symmetry is discussed in graphs for any line of symmetry. However, in algebraic representation line symmetry is discussed only for even functions, those which have line of symmetry in the y-axis.

In the whole work on periodicity the authors did not discuss an example of an oscillatory and non-periodic function. Moreover, the calculation of period only on special points can lead students to ignore the invariance of period when calculated on different points. Thus, introduction to periodicity can lead the students to consider periodicity as: repetition of the special points in the graph, repetition of the trace from 0 to 2π, oscillation in graph where the value repeats even without any regularity, or even a line symmetric graph with vertical line of symmetry.

The pointwise approach taken during all the work with functions can erect a barrier for students in perceiving properties which involve variation. Therefore, for these properties the students can opt for the rules of recognition such as direction of the graphs. The teacher does not emphasise this sort of rule but the textbook is full of these rules.

7 Summary

The concept of function is first introduced to Brazilian students in the eighth grade of primary schooling after working with first and second degree polynomials. The Cartesian representation is used as a visual feedback representation while almost all the actions are made in the algebraic representation. Tables are used as an auxiliary representation to plot graphs.

After a general introduction to functions using different types of algebraic functions, the students started to study families of functions in which different properties are studied. The main points of the way students studied the function properties are:

- turning points are treated as special points highlighted in graphs and associated with extreme values and are also used as special points to trace graphs of quadratic, absolute values and trigonometric functions;
- constant functions are briefly studied as a special case of first degree polynomials;
- monotonicity is first mentioned in the general introduction to a wide set of functions but this property receives emphasis later, only for the linear functions. For linear functions, it is illustrated by graphs, linear coefficients, and tables;

- derivative is introduced again restricted to linear functions by the formula to calculate the rate of average change and is linked to the linear coefficient. Nonetheless, the teacher himself introduced the connection between the 'rate...' and inclination of graph;
- second derivative is marginally explored by tracing parabolas with different curvatures;
- range is introduced in all the families of functions. Nonetheless, this concept is really emphasised for functions with turning points. Detailed work is done for sine and cosine functions, which have bounded range;
- line symmetry is presented associated to turning points in a geometrical way. It is also discussed in a functional way when restricted to line symmetry in the y-axis. Nonetheless, it is generalised for line of symmetry different from the y-axis in the Cartesian representation;
- periodicity is introduced in a way that the students do not distinguish a periodic function from other oscillatory function. Also, the period of function is calculated using special points such as turning points and roots.

The way the students are introduced to functions leads me to anticipate the following difficulties, over-generalisations and barriers:

- associations between: parabola and turning point, symmetry and turning points, turning points and extreme values, monotonicity and polarised rules (such as: 'when x is positive, y is negative ...');
- difficulties in relating the different representations of constant functions, in linking the Cartesian and the algebraic representations of line symmetry;
- restriction of perceptions: monotonicity and derivative to linear functions, range to bounded functions;
- exhibition only of a pictorial perception of second derivative;
- over-generalisation of increasing as being positive curvature;
- tendency for adopting polarised rules while perceiving properties and for a pointwise analysis of the properties.

VII — Longitudinal Analysis of the Work of each Pair of Students

This chapter will present the longitudinal analysis of the work of each pair of students divided according to their development in each of the chosen function properties.

1 Description of the pairs of students

Table 1.1 introduces the four pairs of students by attainment levels and sequences of the microworlds.

Table 1.1

Distribution of students in sequence of microworlds per attainment levels

Sequence / Attainment Levels	From FP to DG	From DG to FP
Lower	John & Tanya	Bernard & Charles
Middle	Diana & Gisele	Jane & Anne

Their teacher evaluated the students in attainment levels according to three criteria: scores they obtained in the exams, difficulties they demonstrated in the exercises and participation in the classroom. He said that John & Tanya were students with lower scores in the exams and with difficulties in learning maths, but they worked hard in mathematics classes. Bernard & Charles always had lower scores in the exams, but they had less difficulty in learning maths than John & Tanya. Their teacher attributed Bernard & Charles' failure in the exams to their lack of interest in doing homework and in participating in classroom activities. As regards Jane & Anne and Diana & Gisele, their teacher judged that they were in the middle attainment level in relation to that of their colleagues in the class. However, he distinguished these pairs according to consistency in scores. Diana & Gisele's scores varied from exam to exam while those of Jane & Anne did not. In addition, the teacher evaluated these four students' participation in the classroom as being poor.

Two of the teacher's comments are relevant to this study. Firstly, Jane & Anne had more facility in doing repetitive problems and had difficulties in problems that call for creativity. He explained "they never come with an unusual solution of a problem". Secondly, he affirmed that John & Tanya had considerable difficulty in formalising as well as in working with mathematics conventions. On the other hand, these students were able to understand and to solve contextual problems using common language.

Among all the students, only Bernard, Charles and Tanya had done previous systematic work using computers for word processing. The other students had worked with computers once or twice in English and music classes at school. None of the students had ever worked in mathematics topics with computers.

Regarding their interaction in group work, only Bernard & Charles had never worked together. The other pairs were used to working together in a collaborative way.

2 Bernard & Charles' perceptions of the function properties

Bernard & Charles followed the activities from DG to FP microworlds.

2.1 Turning point

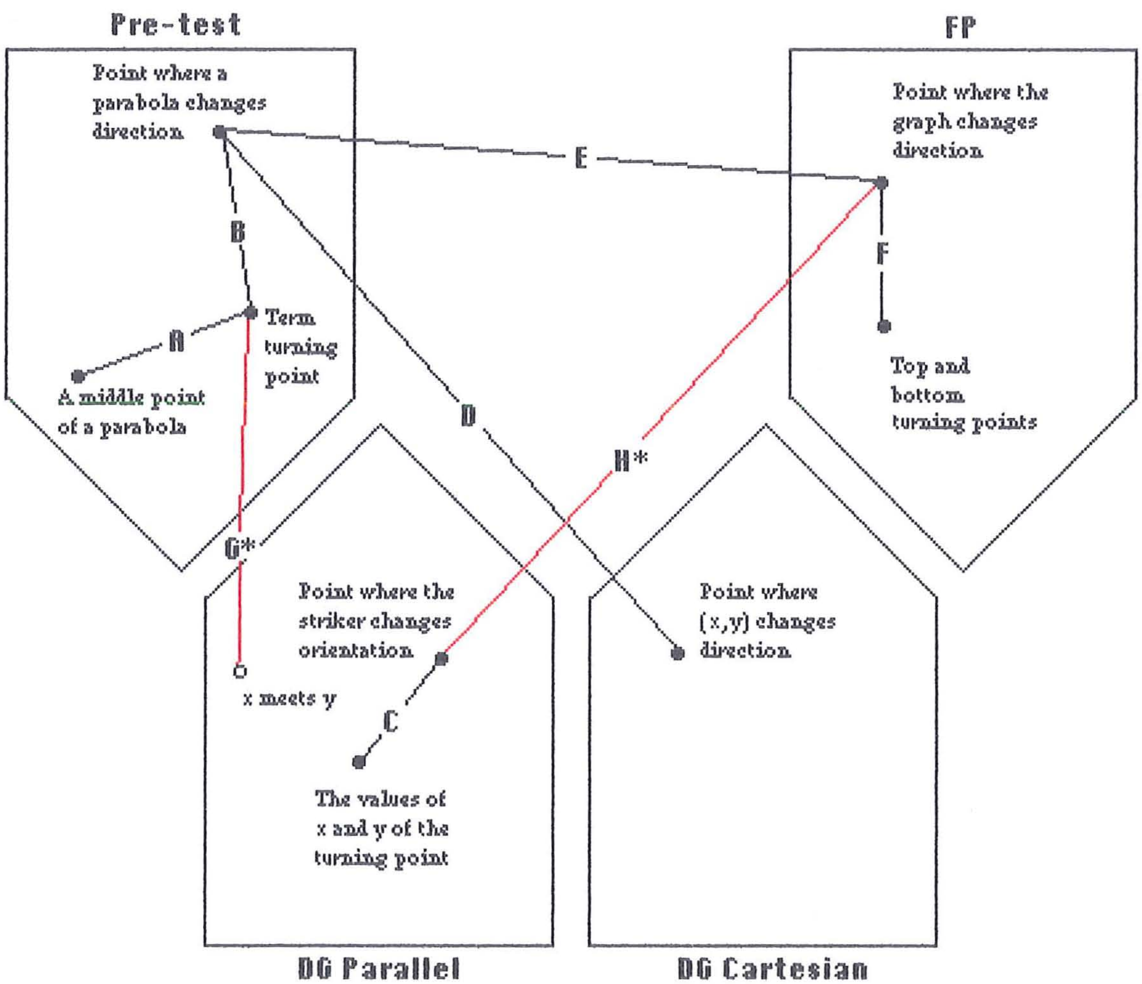
In the pre-test all Bernard & Charles' perceptions of turning point were associated with parabolas. Charles, for example, defined turning point as being 'point where a parabola changes direction' (see link B). Bernard presented the idea of turning point by drawing a parabola with an arrow pointing to the turning point (see link A).

Diagram 2.1 points to a close relation between turning point and parabolas which is evident by the presence of the perception of turning point as being 'point where a parabola changes direction' in all microworlds containing Cartesian representation. This relation shows that Bernard & Charles had a pictorial perception of turning point in Cartesian representation. They also started to call sine graphs 'many parabolas', even through they knew that the graphs were not parabolas.

In DG Parallel, this pair of students characterised the strikers by two kinds of special points: 'point where y meets x', and 'point where a striker changes orientation'. The last one, which corresponds to turning point, was prompted by their observations of the striker given by $y=0.25x^2-8$. Unlike the other chosen parabolas, the turning point of this one is not 'the point where y meets x'. Later, they generalised this idea to the strikers of other parabolas. Therefore, Bernard & Charles constructed a variational perception of turning point in DG Parallel by comparing the behaviour of x and y. Nonetheless, as diagram 2.1 shows, this variational perception was neutralised by the other microworlds in which Cartesian system appears, staying isolated in DG Parallel.

Diagram 2.1

Charles & Bernard's perceptions of turning point



Link D shows that Charles brought the idea of turning point as being 'place where the parabola [(x,y)] changes direction' from their pre-test to DG Cartesian. Despite discriminating turning point in both DG microworlds, Bernard & Charles did not link this idea to the perception of turning point as 'point where the striker changes orientation'. This separation was evident when Charles examined the striker of $y=-0.25x^2$. While looking only at y, he kept repeating that it was not a parabola. He changed his mind only after observing the motion of (x,y).

In FP, as in DG Cartesian, Charles & Bernard identified turning point as 'point where the graph changes direction'. This idea was first presented in their pre-test (see link E). The turning point was also observed by Bernard & Charles as an invariant point after a horizontal stretch of a parabola.

After a vertical translation between the graphs of $y=0.25x^2-8$ and $y=0.25x^2$, Bernard & Charles affirmed that the only thing which changed was the turning points. This remark shows how strong was the use of special points in their perceptions. Another evidence of that is the use of turning point to recognise the shape of a parabola, and also their way of calculating period by the frequency of turning points.

Bernard & Charles had the opportunity to explore turning point from another viewpoint. By searching for characteristics to describe the graph of $y=7\sin(0.125\pi x)$, after a vertical translation from this graph to the one of $y=7\sin(0.125\pi x)+6.9$, Bernard started to distinguish two kinds of turning points: top and bottom ones (see link F). Therefore, Bernard & Charles developed a perception of turning point related to extreme values which was not presented in their pre-test.

In the final interview, while matching the strikers with the graphs, the students connected turning point in graphs to 'the point where x meets y' in strikers (see link G*). Once more Bernard & Charles linked special points in two different representations as having the same meaning. This kind of cannection has similarities with the emphasis of their school knowledge on special points when studying functions.

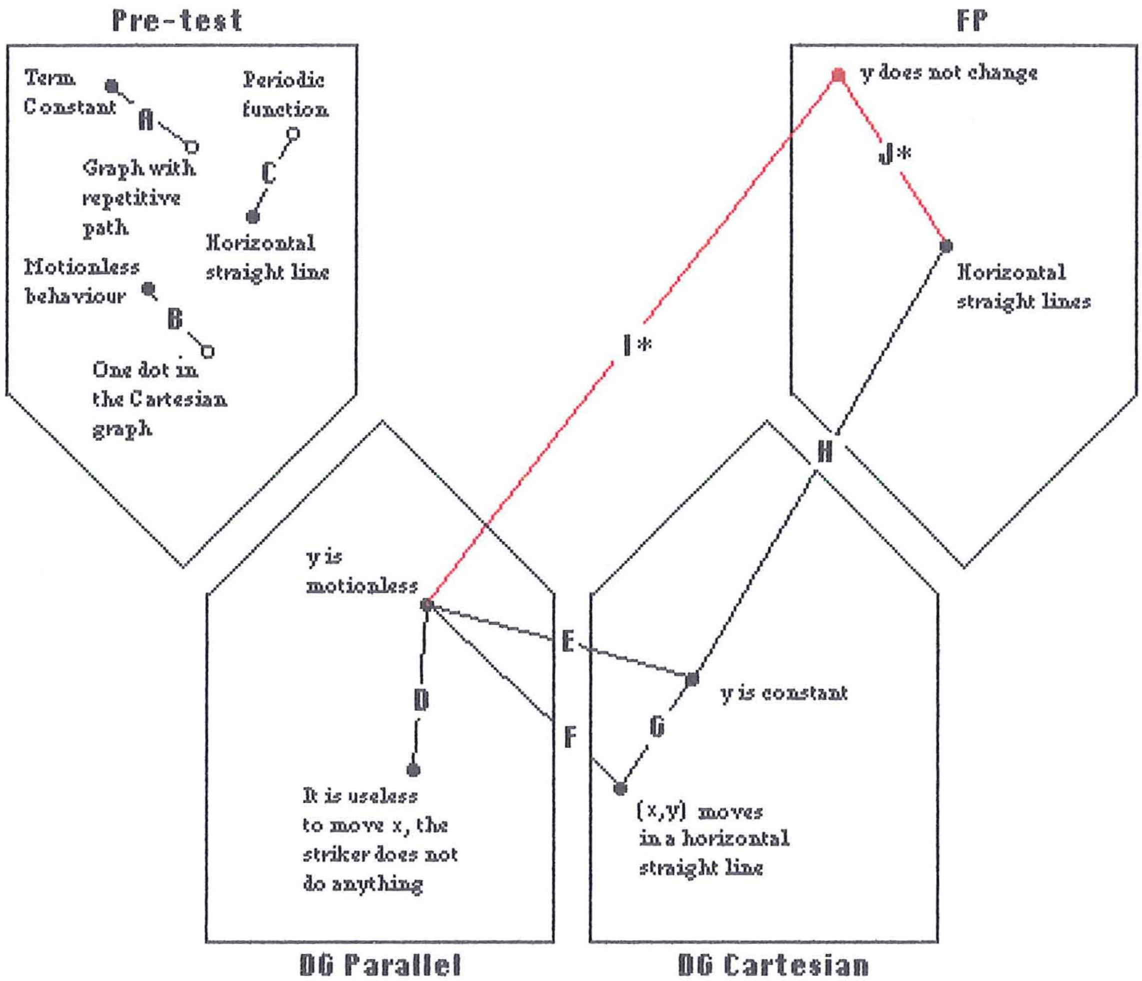
On the other hand, after linking the idea of 'y follows x' in strikers to the idea of increasing in graphs, Bernard & Charles noticed that 'point where a parabola changes direction' should correspond to 'point where the striker arrives and returns' (see link H*) which was also expressed as 'point where the striker changes from 'y follows x' to 'y does not follow x'.

2.2 Constant function

Diagram 2.2 shows two kinds of problem concerning Bernard & Charles' perceptions of constant function in the pre-test. Firstly, both students mismatched the terms constant and periodic (see links A and C). Secondly, Charles, the only student who tried the exercise of tracing a graph from verbal description, represented a motionless car by a dot in a graph of distance versus time (see link B). Although Bernard & Charles' perceptions of constant function in the pre-test were incorrect from a mathematical viewpoint, they constructed a variational perception of constant function in the research environment. Moreover, they used the sequence of microworlds to change their perception of constant function in the Cartesian representation.

Diagram 2.2

Charles & Bernard's perceptions of constant function



In DG Parallel, Bernard & Charles characterised the strikers corresponding to constant functions as being motionless. Thus, these strikers were considered to be completely different from the ones corresponding to linear functions. The pairs of students also perceived constant functions identifying the idea of independence of x. They affirmed that the striker of y=6 was a nonsense striker: “it is useless to move it [x], it [the striker] doesn't do anything” (see link D).

In the first analysis of the striker given by y=6 in DG Cartesian, Bernard confused the idea ‘y is motionless’ from DG Parallel with the idea ‘(x,y) is motionless’. This confusion was the starting point of links E, F and G between ‘y is motionless’ and the fact that ‘(x,y) moves in a horizontal straight line’. Diagram 2.2 suggests that the interaction with DG Cartesian acted as a bridge for the students to connect the variational perception of constant function they constructed in DG Parallel to the

Cartesian representation in FP. The possibility of analysis of the behaviour of x, y and (x,y) as different objects is what made the bridge possible.

The graphs of constant functions were described by Bernard & Charles as horizontal straight lines while working in FP. Moreover, they reported that the lines were horizontal because 'y is constant' (see link H). In my view, this rationality evidences a synthesis between the idea constructed in DG Parallel and their knowledge about graph. Considering their pre-test, this synthesis indicates a great change in their perceptions of constant functions in graphs. Diagram 2.2 suggests that the perceptions constructed by Bernard & Charles in the research environment had supplanted their previous knowledge presented in the pre-test.

As in the research environment, in the final interviews Bernard & Charles connected 'the motionless behaviour of the striker' with 'the horizontal straight line graph'. The important point is the explanation of this connection: "because y does not change" (links I* and J*). This explanation is evidence that the exploration of DG Cartesian really worked as a bridge to the variational view of constant function in the Cartesian representation.

2.3 Monotonicity

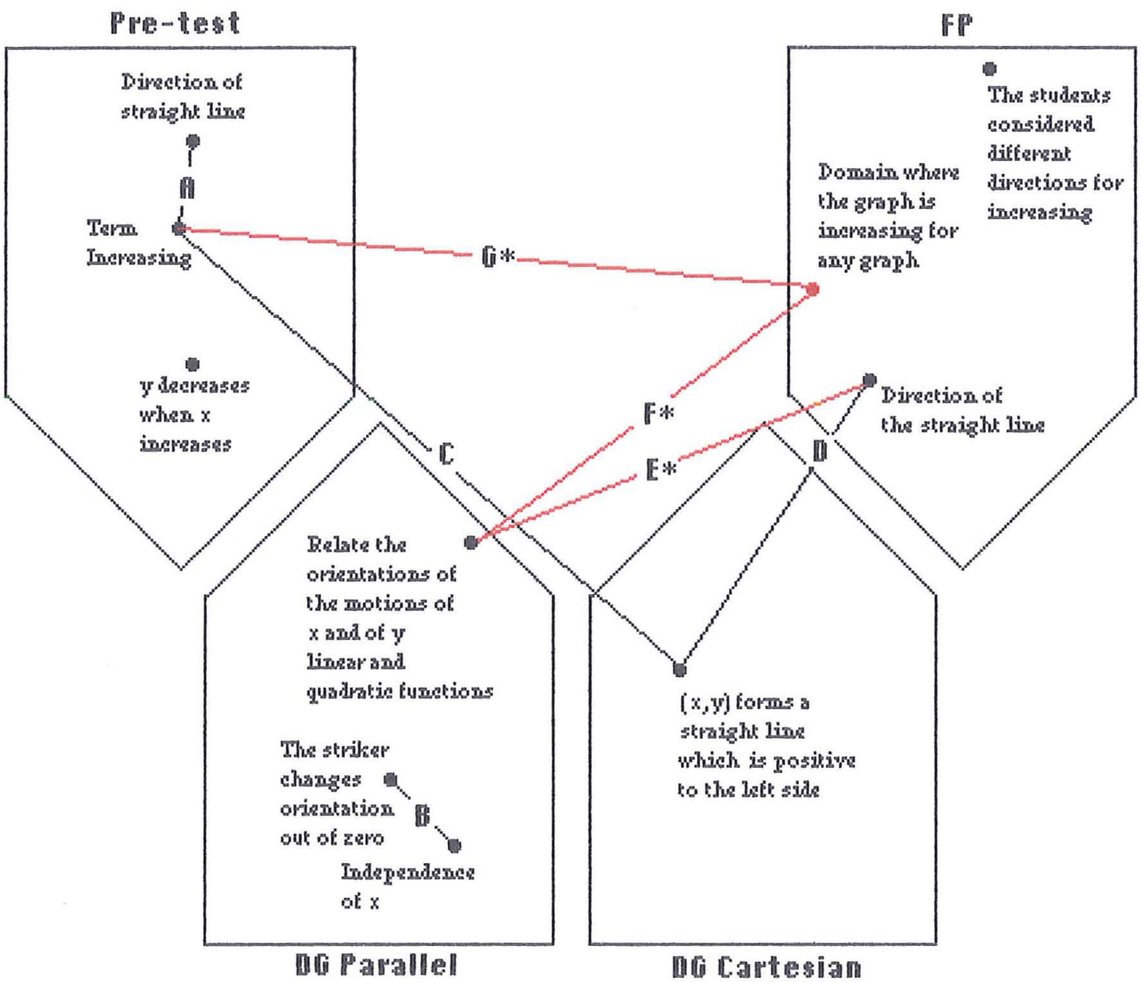
Diagram 2.3 shows that Bernard & Charles used the terms 'increasing' and 'decreasing' only for linear graphs. Moreover, they linked these terms only to the inclination of a straight line and to rules involving positive and negative numbers. This fact seems to be an effect of the school emphasis on this property in studying the family of linear functions. In the pre-test, for example, Charles & Bernard identified the term 'increasing' by 'direction of graph corresponding to linear functions'. In addition, they were not able to determine where the graph of y=3/x was increasing or decreasing (see link A).

Bernard & Charles also discriminated monotonicity in a variational way in their pre-test and in DG Parallel, but these perceptions remained isolated. These perceptions seem to have been triggered by understanding of the term 'increasing'. In the graph of y=3/x Bernard was able to relate the behaviour of y and x without linking it to the term 'increasing'. The students were also able to interpret monotonicity in the pointwise graph. This provides evidence that their previous knowledge about monotonicity can be considered an obstacle for their variational perceptions of this property.

In DG Parallel Bernard & Charles discriminated monotonicity by 'orientation of the motion of y'. Even before knowing that each striker hides a function, they used the above characteristic to describe strikers of linear functions. These students considered whether these strikers follow the same orientation of x or not.

Diagram 2.3

Charles & Bernard's perceptions of monotonicity



Despite being isolated, from a mathematical viewpoint this variational perception was generalisable among other families of function such as parabolas. This perception was used by the students to distinguish strikers of linear functions from those of non-linear functions. For example, the idea of monotonicity was generalised to the striker of $y=0.25x^2$ as 'sometimes it follows one orientation, sometimes it does not'. In this generalisation, Bernard & Charles separated the domain into positive and negative to verify where each striker follows the orientation of x.

Unfortunately, their tendency to separate everything into positive and negative induced Bernard & Charles to think that 'y is independent of x' for strikers of sines. This polarisation represented an obstacle to their generalisation of the above perception of monotonicity to strikers of sines. Moreover, the polarised thinking induced an idea of independence of x for the striker of $y=7\sin(0.125\pi x)$ (see link B). As this striker kept changing the orientation in positive as well as in negative domains, Charles concluded that "it doesn't obey the triangle". It seemed that the students expected that the strikers changed orientation only when x passed at zero.

In DG Cartesian Bernard & Charles explored the property of monotonicity by 'shape formed by the motion of (x,y)' using concepts from their previous knowledge. This property was associated to rules such as 'straight line is positive to the left side' when the students tried to explain why the striker of $y=-x$ was decreasing (see link C). There was no evidence of link between the perception of monotonicity discriminated in both DG. Further evidence that the students brought this perception from their previous knowledge is its limited application for strikers of linear functions. Moreover, while Charles was examining this perception in the striker given by $y=0.25x^2$, he abandoned the verification as soon as he realised it was a parabola. Therefore DG Cartesian did not create a spontaneous bridge between DG Parallel and the Cartesian system for this pair of students in the property of monotonicity.

In FP, the idea of monotonicity was discriminated by Bernard & Charles as 'direction of the straight line' and this also was influenced by their previous knowledge. 'Direction of the graph' was classified into two types: increasing direction and decreasing direction (see link D). For example, after a horizontal stretch between the graphs of $y=-x$ and $y=-(1/4)x$, Bernard argued that both graphs had the same direction. This division can be considered as a polarisation in their understanding when the slope was not considered.

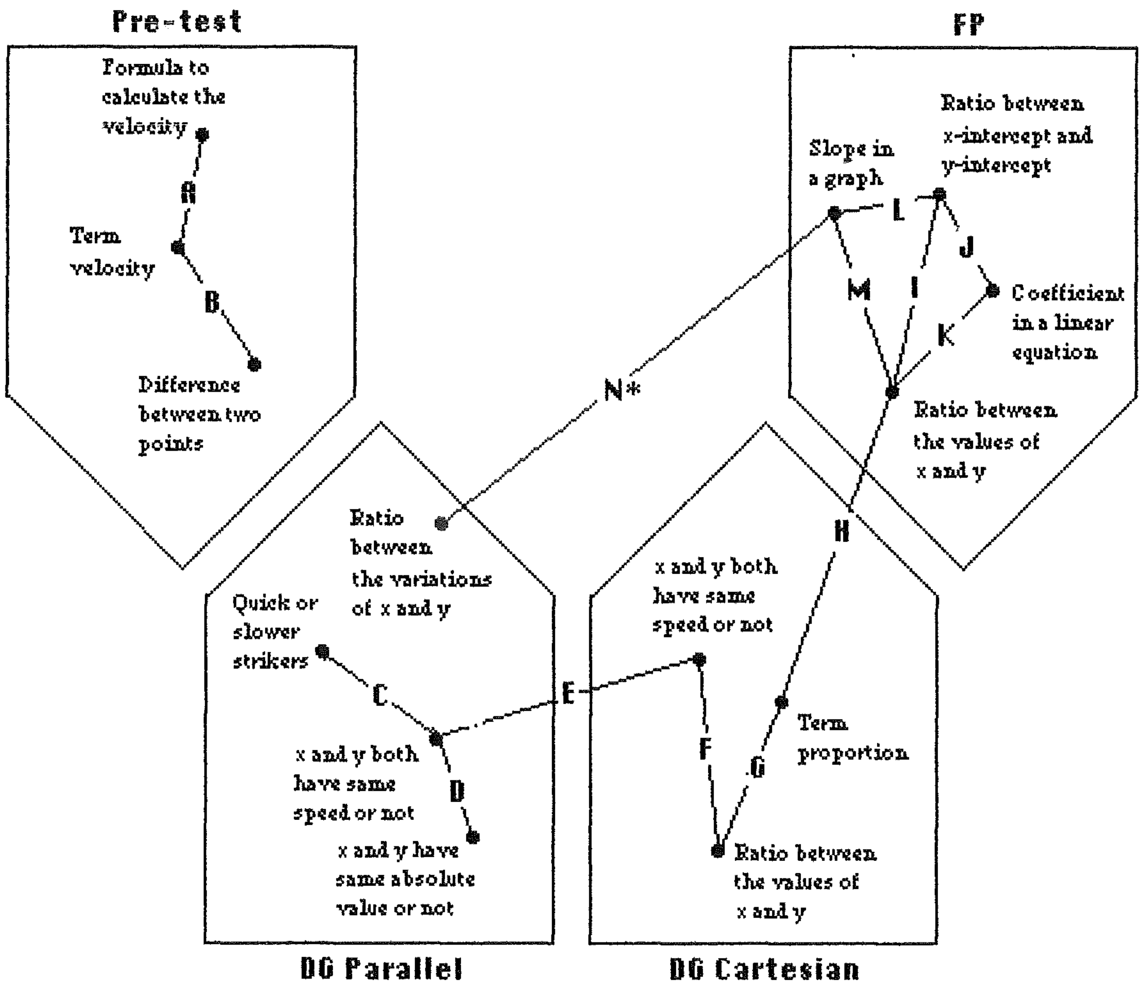
The interaction with dynamic transformations of graphs in FP prompted Bernard & Charles to see an order in the idea of monotonicity. They realised the connection between monotonicity and derivative. While investigating the idea of increasing, a horizontal stretch in the graph of $y=x$ encouraged the students to connect 'direction of straight lines' and 'slopes'. For instance, Charles argued that the change from increasing to decreasing depends on where you have the graph. He explained that anyway the command changes the direction of the graph but it can pass from one type to the other type.

Bernard & Charles' perceptions of monotonicity seemed to be a great discovery for them. In the final interview they brought the generalisation from DG Parallel to the Cartesian system, but it was not straightforward. Firstly, they linked the term increasing with 'direction of the graph' to 'y follows x' for increasing and 'y does not follow x' for decreasing (see link E*) limited to linear functions. As they had this perception of 'y follows x' or 'y does not follow x' for strikers given by parabolas, they brought back the link. Charles explained "when it [y] does not follow x, the graph has this direction, in the middle [of the graph], it changes direction". Moreover, they used for those directions the terms 'increasing' and 'decreasing' (see links F* and G*). This seems a great strength in their perception of the property because it allowed them to overcome the obstacles created by using the terms 'increasing' and 'decreasing'.

2.4 Derivative

Diagram 2.4

Charles & Bernard's perceptions of derivative



In the pre-test, Charles & Bernard used only pointwise views to discriminate derivative. For example, they knew the formula for velocity (see link A) but they did not know how to use it. Moreover, they did not link velocity to the coefficient in an equation of linear function. As regards the use of graphs to interpret derivative, both students only interpreted the idea in discrete graphs. It seems that they discriminated the derivative by the difference between two points (see link B). The slope was not linked by them to the idea.

Diagram 2.4 shows that Bernard & Charles developed a variational view of the notion of derivative in a continuous process throughout the research environment. They started comparing the speed of different strikers. Later, they constructed a ratio to measure the speed of a striker. Finally, in FP they brought this ratio to link with their perception of derivative in other representations (see link H). Therefore, DG Cartesian was used as a bridge for this variational perception from DG Parallel to the Cartesian System. Nonetheless, the ratio created by Bernard & Charles was based on 'linear' functions[1], as they considered the absolute values of x and y, instead of their variation. Bernard & Charles seemed to know the definition and how to calculate the derivative as velocity since the pre-test. Nonetheless, as Diagram 2.4 shows, their development seems to have blocked these previous ideas. It is interesting to observe that despite knowing the formula for calculating velocity as $\Delta y/\Delta x$, they did not use or mention this formula while working with ratio in DG Parallel, nor did they consider variations of x and y. The students moved from 'ratio of absolute values of x and y' to 'ratio of variations of x and y' only in the final interview.

Bernard & Charles constructed the idea of derivative in DG Parallel in two steps. In the starting activity with DG Parallel they classified the strikers as slow and fast. Then, to describe the strikers in the following sessions, the perception of slow or fast was replaced by a comparison between the speeds of y and x (see link C). For example, to describe the striker of $y=x$, Bernard said that it had the same speed as x. The other strikers with different speeds of y and x were characterised as 'y is quicker than x'.

Bernard & Charles' construction of the perception of derivative as speed was not straightforward in DG Parallel. They associated 'same speed' and 'same absolute value' while describing the strikers given by $y=x$ and $y=-x$ (see link D). They recognised this association when analysing the striker of $y=x-6$. Despite that, Charles returned to it when analysing the striker of $y=2x$. This association became more salient when the students compared the speed of the strikers corresponding to $y=0.25x^2$, $y=0.5x^2$ and $y=0.25x^2-8$ in the positive domain. They were trying to verify which striker was quicker by observing which striker was ahead of the others. This comparison led the students to use the idea of infinity from previous knowledge in order to overcome the association. Also, by realising that the striker of

[1] Here I am using 'linear' meaning that it was not an affine function. In other words, it is like $y=ax$.

$y=0.25x^2-8$ started behind the others — when x was at zero — Bernard argued that having the same speed, this striker could not disappear at the same time as the striker of $y=0.5x^2$.

The idea of derivative as speed built in DG Parallel was strengthened by Bernard & Charles in DG Cartesian. They calculated derivative as 'ratio between the values of x and y' for some strikers of linear functions (see links E and F) using the term 'proportion' to denominate this ratio (see link G). The idea of speed was also used to distinguish the two strikers of sines. Unfortunately, I had no evidence that this perception was linked to the idea of derivative as slope while exploring DG Cartesian.

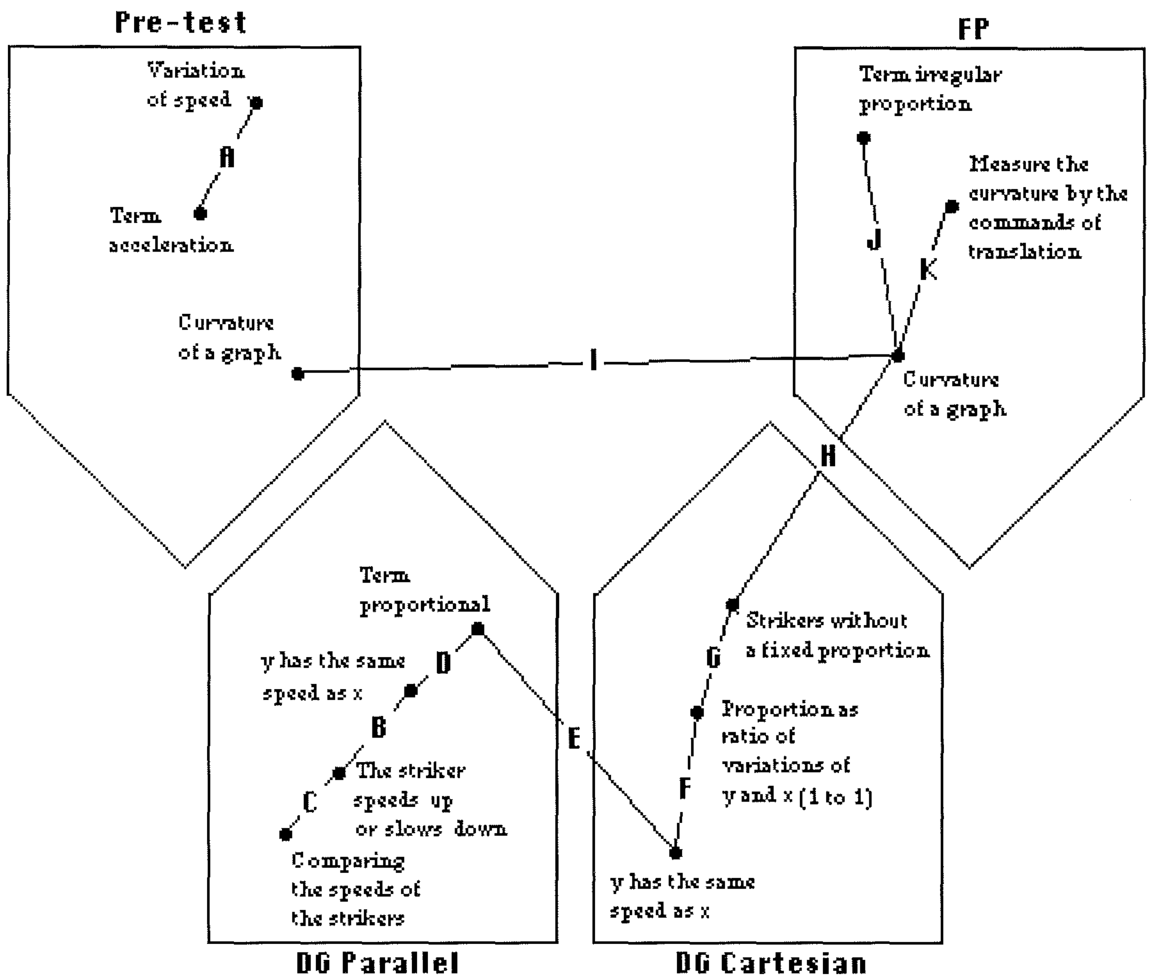
In FP by investigating 'the ratio between the values of x and y' while exploring the dynamic transformations of graphs, these students linked this idea to coefficient in equation as well as to slope in graph. First, Charles used the point indicator icon to verify if the derivatives of the graphs given by y=abs(x) and y=abs(x)-10 were the same. Second, he linked 'ratio between the values of x and y' to 'linear coefficient in a equation' (see link K). He was investigating the idea by a horizontal stretch between the graphs of y=2x and y=x. Up to this point, the students did not generalise the above-mentioned perception of derivative to affine functions. The generalisation happened in two steps. First, Charles noticed that while translating the graph of y=2x vertically, 'the ratio between x-intercept and y-intercept' stayed the same (see links I and J). In a second step, by the parallelism between the graphs of y=x and y=x-6 as well as by comparing them to the behaviour of the strikers corresponding to these equations, Charles & Bernard concluded that 'the ratios between the values of x and y' should be the same (see link L). This passage can be considered a beginning of link M between slope and the idea they had of derivative, which was concluded while Charles was exploring the horizontal stretch in the graph of y=x. He argued that 'the proportion is what provokes the inclination'.

Despite generalising their perception of derivative as 'the ratio between the values of x and y' to affine functions in FP, Bernard & Charles did not perceive the incompatibility in the way they measured this ratio. They reviewed the link between the 'ratio between the values of x and y' and 'inclination of linear graphs' in the final interview. They started to calculate 'ratio...' by comparing 'steps that y moves while x moves one step' (see link N*). Moreover on matching the strikers of y=x and y=x-6 to their graphs, they explained that "the difference of the strikers should be 6 steps because of the difference between the y-intercepts".

2.5 Second Derivative

Diagram 2.5

Charles & Bernard's perceptions of second derivative



In the pre-test Bernard & Charles discriminated second derivative as 'variation of speed' (see link A) and as 'curvature of a graph' without linking these perceptions. They traced the graph of distance per time of a car as a straight line for constant as well as for variable speed. Moreover, they did not use 'curvature' of a parabola to interpret acceleration. On the other hand, they were able to distinguish graphs of parabolas by their curvatures. Also, measuring curvature was a difficult task. They affirmed that two graphs of two parabolas translated vertically had different curvature.

Bernard & Charles presented a continuous and connected process for second derivative, as they did for derivative. They constructed a variational view of second derivative while trying to calculate 'ratio between values of x and y' on strikers of non-linear functions in DG Parallel. Then, they used DG Cartesian to strengthen this perception by building the idea of variable 'ratio between values of x and y'. Finally, in FP they linked the curvature of a graph with 'absence of a fixed ratio', which they called proportion.

It is interesting that Bernard & Charles used the idea of variable speed in the starting activity with DG Parallel while referring to the speed of the striker given by $y=0.25x^2-8$. However, on formalising the idea to strikers of parabolas, they characterised the speed of this striker as "it is faster than x".

It was only on comparing the strikers of y=x, y=2x and $y=0.25x^2$ that Charles & Bernard started building the idea of constant speed. As the striker of y=2x started ahead of the striker of $y=0.25x^2$ and as the first striker was overtook by the second one, the students concluded that the last striker accelerated to become quicker than the striker of y=2x. Nonetheless, they thought that the striker of y=2x slowed down (see link C). Therefore, they assigned the idea of 'constant speed' only to the striker of y=x, which has the same speed as x (see link B).

The episode discussed above revealed to me the students' association between the idea of 'being quicker than...' and the idea of 'being accelerated'. This association was made clearer by Charles' observation of the striker given by y=2x. As he noticed that 'y overtakes x', he concluded the "striker [y=2x] is more accelerated than the triangle [x]".

The idea 'the striker has the same speed as x', which was called by these students 'proportional' (see links D and E), also appeared in DG Cartesian as an important step in their construction of the idea of constant and variable derivatives. This idea was constructed by: their observation that 'y moves with same step as x' for the striker of y=x; the possibility of calculating 'ratio between the values of x and y' as a way to generalise the idea to the striker of y=2x (see link F); and the impossibility of calculating this ratio while comparing the strikers of $y=0.25x^2-8$, $y=0.25x^2$ and y=2x. They concluded that these strikers had not a constant derivative (see link G).

As Bernard & Charles had constructed the idea of 'variable derivative' as "there is no fixed proportion" in DG Cartesian to strikers of parabolas, in FP they linked this perception to 'curvature' of a parabola (see link H). After trying to distinguish the graphs of $y=0.5x^2$ and $y=0.25x^2$, Charles made the link. They called that characteristic 'irregular proportion' (see link J).

Regarding their difficulty in measuring of curvature, it also appeared in FP. Nonetheless, by using the vertical translation while exploring the idea of curvature, the students realised that the curvature could not be measured only by 'distance between two symmetrical points' (see link K). Moreover, they realised that they needed another point to determine a parabola, in particular to distiguish 'curvature' of parabolas. In addition, Bernard developed a method of verifying whether two parabolas had the same curvature by using the vertical translation of FP. Bernard reported that the graphs of $y=0.5x^2$ and $y=0.5x^2-10$ had the same curvature because the command used did not alter it.

In the final interview Bernard & Charles only confirmed the link they made in the research environment. On being asked which was the corresponding idea for curvature of graph in strikers, they linked it to 'absence of a regular proportion' meaning 'absence of a fixed ratio between the variations of x and y'. Nonetheless, it was not straightforward. Firstly, they argued that they recognised a graph with curvature by existence of turning point. By covering the part of the graphs which contained the turning point, I asked if they could decide which graph was a straight line and which was a curve. They answered "Of course!". On being asked if they could distinguish between two strikers which was a parabola and which was a straight line, they asked to place both strikers in DynaGraph in order that they could try. I moved them taking care not to pass through the turning points. By having constructed the idea that strikers of linear graphs had a 'fixed ratio between the variations of x and y', the students observed that there was one which had not a fixed ratio concluding the link.

There is an interesting point to consider while analysing the students' perceptions of second derivative. While matching the strikers and graphs of $y=0.25x^2$ and $y=0.5x^2$, Bernard & Charles corresponded 'speed of strikers' to 'curvature'. In general, they expressed it as 'it is more closed or more opened'. I am not sure that, on interpreting graphs with curvature, the students distinguished curvature from slope.

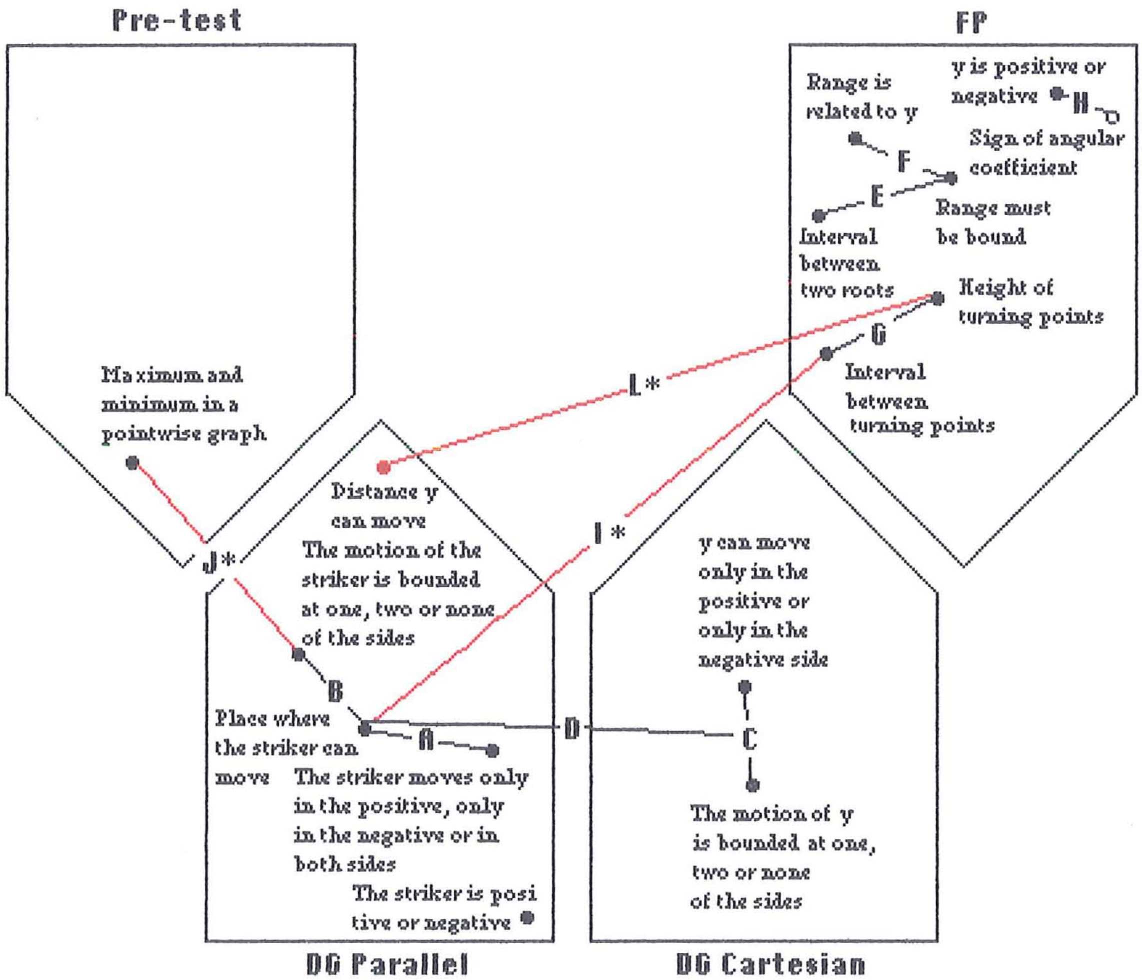
2.6 Range

In the pre-test Bernard & Charles demonstrated no familiarity with the term 'range'. For instance, they did not answer any question about range. They only identified extreme values for the discrete graph.

From the starting activity with DG Parallel Bernard & Charles explored range as 'place where the strikers of $y=0.25x^2-8$ and $y=7\sin(0.25\pi x)$ can move', which was motivated by their need to move the strikers to score in DG Game.

Diagram 2.6

Charles & Bernard's perceptions of range



While describing the strikers in DG Parallel, Bernard & Charles started from a polarised approach to range and moved into one that involves the idea of bounded and boundless range. This development was motivated by their need to generalise their perception of range to all the strikers, joining the strikers by similar range. Thus, the activities designed for the research where the students need to classify as well as compare the strikers led Charles & Bernard to abandon the polarised approach. Starting by describing the striker of $y=-0.25x^2$ as 'it moves only in the negative side', they generalised this perception to compare it with the striker of $y=0.5x^2$ as "the striker moves only in one of the sides". Their first attempt to overcome the limitations of the polarised perception of range was the characterisation 'bounded motion of the striker of y=7sin(0.125 x)', which was described as "it does not go to the corner of the screen". Following this characterisation, the students generalised the idea to the strikers corresponding to y=2x and y=x as 'they can move all the screen'. In fact, their polarised perception of range was abandoned by Bernard when observing similarities between the strikers of $y=0.25x^2-8$, $y=-0.25x^2$ and $y=0.5x^2$. They generalised the perception to the striker of $y=0.25x^2-8$ as "the striker does not go to the end of the axis" (see link B) which was used to classify the strikers, excepting the strikers of constant function that continued to be characterised by a polarised way. This suggests that these perceptions were very close to motion.

The perception of range was imported by Bernard & Charles from DG Parallel to DG Cartesian (see link D). They continued using y to identify range in DG Cartesian. Bernard described the striker of $y=7\sin(0.25\pi x)$ as 'moving half of the axis' for example. From a mathematical viewpoint, Bernard & Charles' perceptions of range in these microworlds were important for the identification of the variables.

In DG Cartesian their polarised perception of range appeared again in Bernard & Charles' work. While describing the strikers of $y=-0.25x^2$, $y=0.5x^2$ and $y=0.25x^2$, they used 'the striker is only positive' or 'it is only negative'. These students discriminated the range of the striker of $y=0.25x^2-8$ as "moving all the y-axis (positive and negative)". As in DG Parallel, in DG Cartesian the approach of bounded range developed by Charles & Bernard allowed them to see ranges of different parabolas as being similar, which did not happen until its generalisation to all the strikers with motion.

The limitations of the polarised perception were overcome when they tried to classify the strikers. Bernard & Charles used bounded or boundless range to join the strikers of linear functions, as well as to join strikers of sines and quadratic functions. Bernard separated range of the strikers of parabolas from that of sines affirming that they go up to infinity in one side. The students also added that the strikers of sines "you can mark [localise extremes], the other strikers are infinity" (see link C).

Compared to its importance in DG microworlds, the idea of range lost strength in the students' characterisation of the functions in FP. There, range was discriminated by Bernard & Charles only in a polarised way. For example, translating the graph of $y=6$ to $y=-3$, Charles classified the graphs of $y=6$ and $y=-3$ observing that their signs of range were positive and negative respectively. Another example of this was Charles' association between negative angular coefficient of $y=-0.5x^2$ and the negative range.

Note that Bernard & Charles' perceptions in both DG microworlds were not linked to the term 'range' presented in their mathematics class. The first time that the students used the term 'range' was in FP while trying to define it in the graph of $y=0.25x^2-8$. It seemed that they had restricted the term 'range' to bounded range (see link E). The association was evident when Charles tried to discriminate the range of this graph as 'the interval between the roots'. He thought strange that Bernard said that range was related to y, not to x (see link F), then, he exclaimed: "but the parabola is infinity".

FP was used by Bernard & Charles as a way to try out their beliefs such as 'range must be bounded', 'range is related to y'. In other words, the exploration of different perceptions while transforming graphs enabled Bernard & Charles to generate examples and counter-examples which motivated discussions. For example, on trying to characterise the graph of $y=7\sin(0.125\pi x)$, the exploration of horizontal and vertical stretches on its graph led them to distinguish two ideas related to range: amplitude as being 'distance between top and bottom turning points' and range as being 'the interval given by the value of these turning points' (see link G).

Nonetheless, their tendency to link objects rather than meaning led Bernard & Charles to associate 'the sign of angular coefficient of quadratic equations' with 'positive or negative range' of the graphs through 'y is positive'. Although Bernard did not associate both perceptions, Charles did (see link H). After a vertical translation from $y=0.25x^2-8$ to $y=0.25x^2$ to guess Bernard's description, Charles realised the association concluding that 'positive or negative range' was not linked to 'positive curvature' of parabola.

The perceptions of range of Bernard & Charles formed two groups: one group constituted by the pre-test and by FP, and the second group constituted by DG Parallel and DG Cartesian. These groups of perceptions stayed completely separated up to the final interview.

In the final interview while matching the strikers with the graphs, the students connected range to 'place where y can move' to identify the family of functions to which a striker belongs. At this time, they used the polarised approach, but on being asked about the corresponding idea of 'bound of the motion of y', the students identified with their previous idea of extreme values (see link J*), which was presented when they distinguished turning point in top or bottom (see diagram 2.1) and presented in their pre-test restricted to discrete graphs.

What is interesting is that the students started to interpret a graph in a variational way to verify its range. For example, on being asked what would happen with a striker of y=7sin(0.25πx) if translating its graph to the one of y=7sin(0.25πx)+6, Charles explained that "the place where y moves would be translated" (see link L*). In addition, he argued "but the length that it moves would be the same" (see link l*).

2.7 Symmetry

Diagram 2.7

Charles & Bernard's perceptions of symmetry

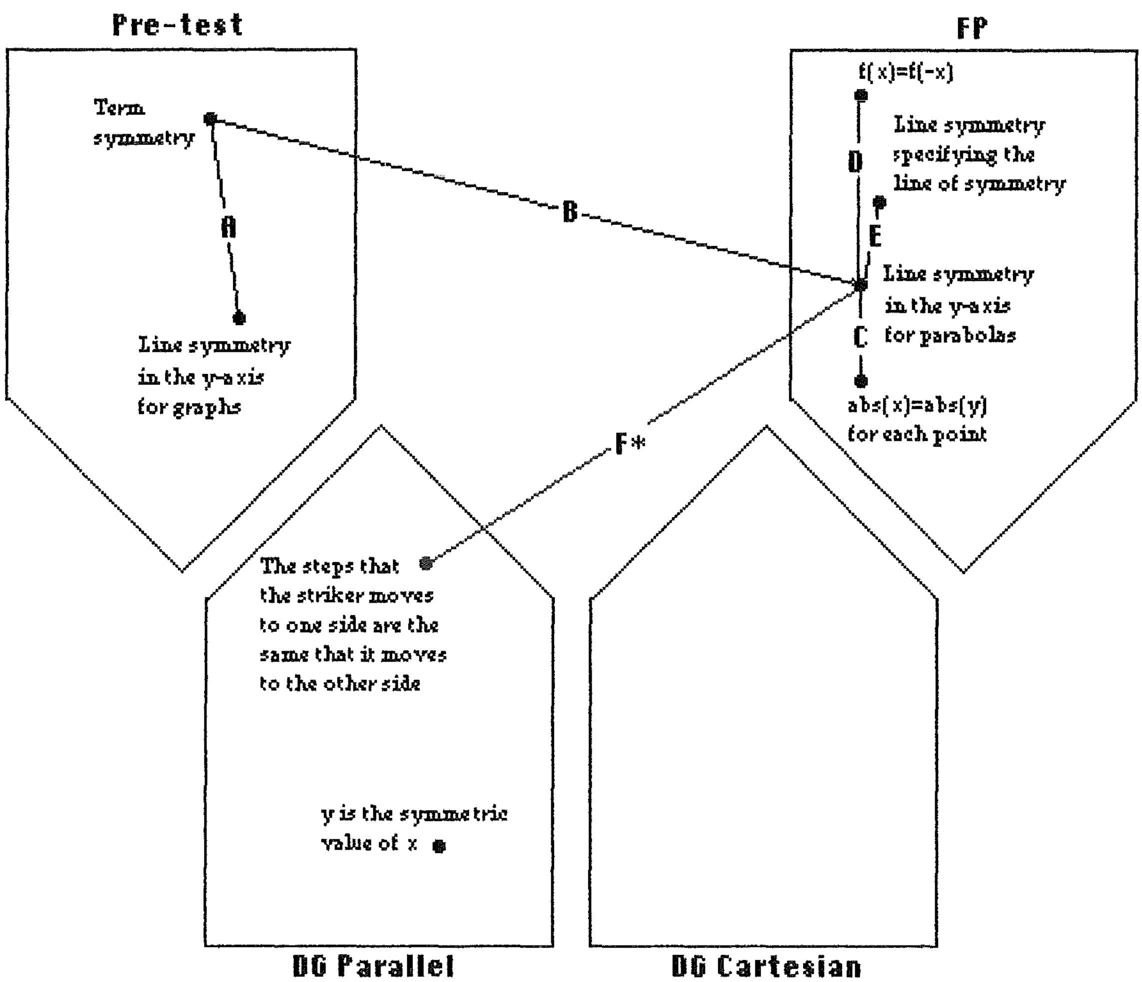


Diagram 2.7 shows that Bernard & Charles's previous perception of line symmetry was pictorial. They used shape of parabolas to discriminate symmetric graphs. In the pre-test, these students identified as being line symmetric only graphs with line of symmetry in the y-axis. The graph of y=5sin(x), for example, was not considered to be symmetric (see link A). Link B shows that Bernard & Charles used this

perception in FP generalising it to graphs with line of symmetry different from the y-axis.

The only perception of symmetry spontaneously discriminated by Bernard & Charles in DG Parallel were in terms of symmetric numbers. For instance, Charles characterised the striker of y=-x as being symmetric because y was always the symmetric number of x. In some ways, this perception is reflected in their belief that line symmetry in parabola always means $f(x)=f(-x)$ which appeared in FP. As regards DG Cartesian, Bernard & Charles did not refer to any sort of symmetry, even to symmetric numbers. Moreover, they used the term line of symmetry in the parabolas associated with turning point.

In FP, Bernard & Charles were also encouraged to seek a pointwise correspondence of their pictorial perception of line symmetry. When trying to make sense of line symmetry as a relation between x and y, the only perception mentioned by the students was 'abs(y)=abs(x) for each point' (see link C). This perception corresponds to the idea of symmetric numbers discriminated by them in DG Parallel, instead of line symmetry. While searching for a new graph to be described by translating the graph of y=abs(x) vertically, Charles revised this perception expressing it as 'abs(y)=abs(x) in both graphs'. So, up to this point, they were able to identify the line symmetry in graphs without making sense of it in a pointwise way.

On trying to compare the graphs of y=x and y=-x while stretching them vertically, Bernard & Charles were able to build a pointwise correspondence to their perception of line symmetry in the y-axis between two graphs (see link D). They argued that these graphs were 'contrary' and verified that $f_1(-x)=f_2(x)$[2] using the point indicator icon. They also generalised this perception to parabolas. They explained the line symmetry in the y-axis in a pointwise way: "A dot here [$f(x)$] must correspond [be equal] to a dot here [$f(-x)$]. All parabolas must be [symmetric]...".

Bernard & Charles' explorations of FP triggered off opportunities to generate counter-examples of associations they themselves generated from particular examples — in general emphasised in school mathematics. For instance, the belief that line symmetry means $f(x)=f(-x)$ and that all parabolas are symmetric enabled the students to generate a critical moment for overcoming the limitation of this pointwise perception of line symmetry. By translating the graph of $y=0.5x^2$ horizontally, they started identifying line of symmetry in all parabolas (see link E).

[2] Here, I am denoting $f_1(x)=x$ and $f_2(x)=-x$.

Nonetheless, Bernard & Charles were not able to reformulate their pointwise perception of line symmetry.

Link F* shows that in a motivated synthesis Bernard & Charles were able to discriminate a variational perception of line symmetry in DG Parallel. Being asked to correspond line symmetry in DG Parallel, the students sought a perception that depends only on the relation of x and y. Unfortunately, their explanation was restricted to parabolas with turning point at (0,0). Charles explained that "the steps [of y] are the same to one side [of x] and to the other [side of x]".

2.8 Periodicity

Diagram 2.8

Charles & Bernard's perceptions of periodicity

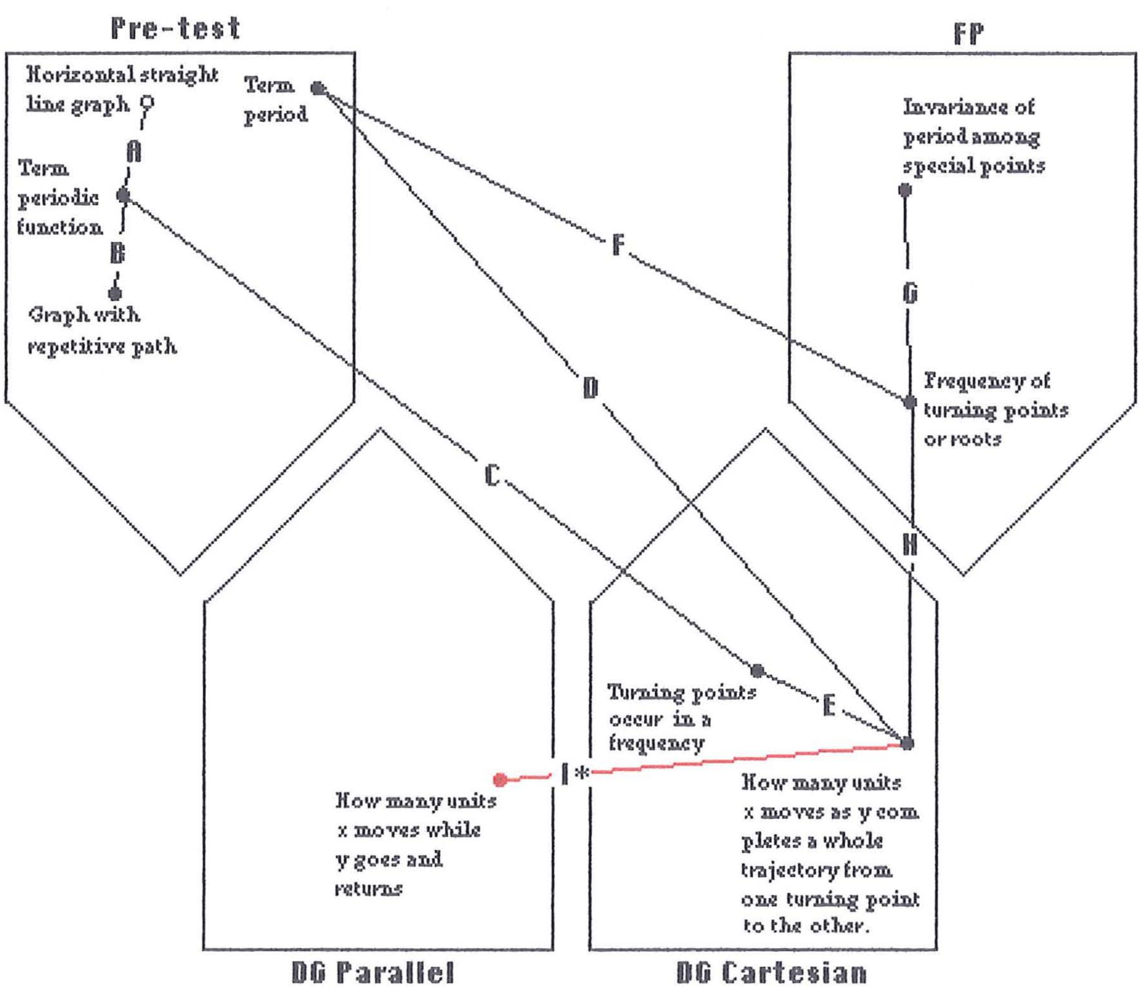


Diagram 2.8 demonstrates that in the pre-test Bernard & Charles mismatched the meaning of the terms constant and periodicity. Moreover, Bernard identified periodic graphs in two different ways: when traced by himself, straight lines were considered

the periodic ones (see link A); and when given in the pre-test — sines, oscillatory graph and a parabola — repetitive graphs were seen as being periodic (see link B). Therefore, they mismatched only the term.

Despite comparing the strikers of sines with different period, Bernard & Charles did not talk about any property related to periodicity in DG Parallel. Instead, they interpreted the periodic behaviour of these strikers as "y does not obey x" (see link B in diagram 2.3). Unlike in DG Parallel, in DG Cartesian Bernard & Charles used periodicity to distinguish the two strikers of sines. This was motivated by their identification of different frequencies of turning points in the shape traced by (x,y) (see link C). After noticing the periodicity of the turning points, Bernard & Charles sought the meaning of the term 'period' in DG Cartesian (see link D). Charles calculated the period counting 'how many units x must move while y makes a complete trajectory' (see link E). The contrast between 'absence of shape' and the motion of x, y and (x,y) led Bernard & Charles to try a variational correspondence for different ideas they had acquired at school.

As in DG Cartesian, in FP the first idea of periodicity discriminated by Charles & Bernard was the frequency of roots and turning points (see link H). Note that this idea was as yet exclusive of special points. That is, they did not perceive that this frequency is invariant at any point they chose. On exploring FP, Bernard & Charles created a critical moment to recognise the invariance of period among special points. They discovered that the period is invariant by the point you could choose to start counting among 'special points' only. For example, the measurement of the frequency based on the top turning points would be the same as that based on bottom turning points. This invariance was the object of one question from Charles who answered by counting it himself (see link G).

The term 'period' was brought by Charles from previous knowledge to make sense in FP (see link F). On exploring periodicity by a vertical translation from the graph of $y=7\sin(0.125\pi x)$ to the graph of $y=7\sin(0.125\pi x)+6.9$, Bernard noticed that both graphs had same frequency of bottom turning points. At this point Bernard linked the frequency of turning points to the term 'period'.

It seems to be important that all these perceptions Bernard & Charles constructed about periodicity were linked to their previous knowledge. Moreover, their previous knowledge informed their understanding and discussion in the research environment. In the final interview Bernard & Charles linked period, which they calculated in graphs in FP, to "how many units x moves while y goes and returns" (see link I*) in

DG Parallel. Thus, they brought back the variational perception that was constructed by themselves in DG Cartesian to DG Parallel.

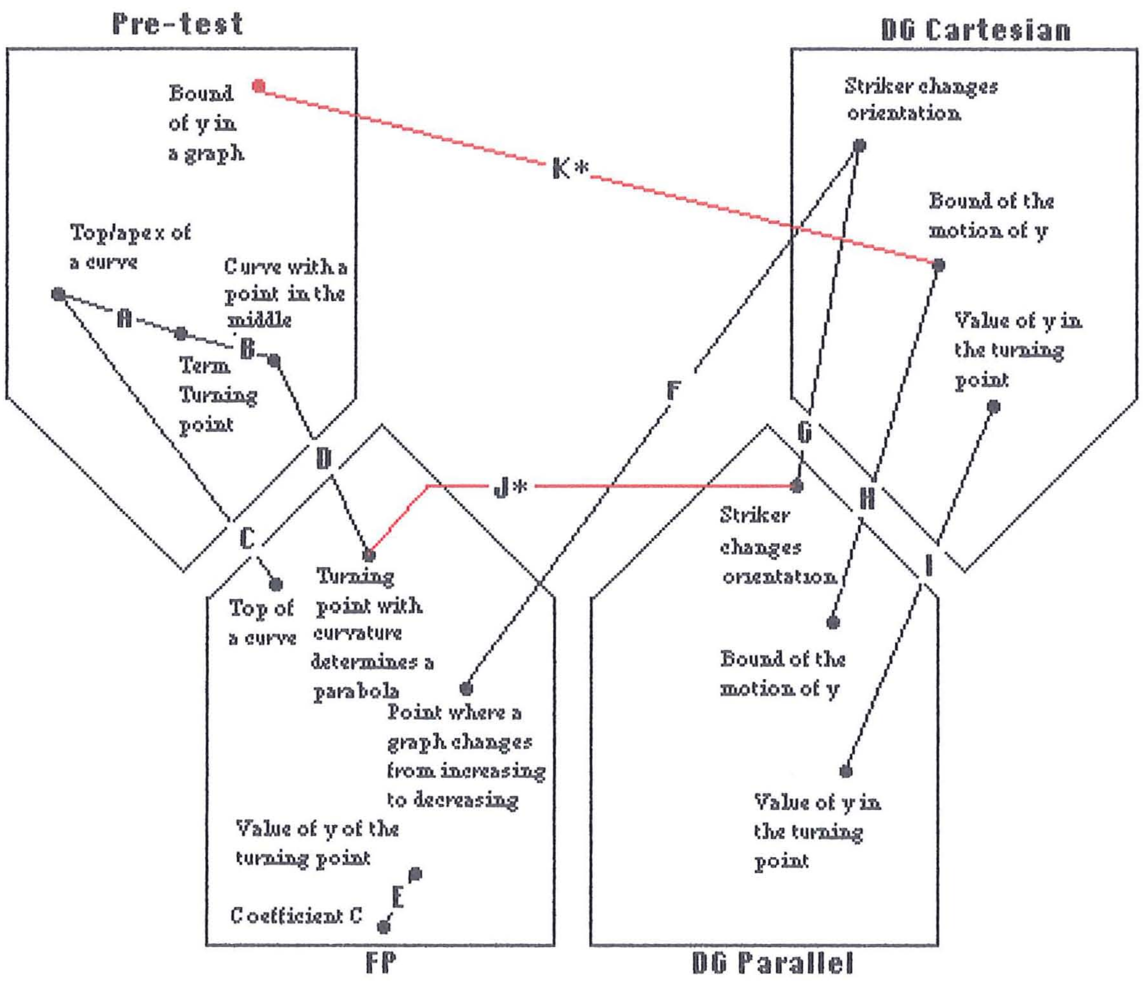
3 John & Tanya's perceptions of the function properties

John & Tanya were one of the pairs of students who followed the activities from FP to DG microworlds.

3.1 Turning point

Diagram 3.1

John & Tanya's perceptions of turning point



As links A and B show, in the pre-test Tanya & John expressed turning point in different ways: 'the top of a curve' for Tanya and 'curve with a point in the middle' for John.

Diagram 3.1 demonstrates a shift in John & Tanya's perceptions of turning point from a pictorial perception in the pre-test to a variational perception in the research environment.

In FP John & Tanya's perceptions of turning point depended on the command used as well as the topic they were investigating. For example, John argued that turning point was 'point where the graph changes from increasing to decreasing' while translating the graph of y=abs(x) into the one of y=abs(x-10) and looking at the point where the graph changes slope. Afterwards, John discriminated turning point of the graph of $y=-0.25x^2$ as the 'top of this parabola' while exploring a vertical translation in this graph and searching for properties to describe it.

By using their previous experiences with the transformations of graphs, John & Tanya linked 'the value of the turning point' to 'the coefficient 8 in the equation of $y=0.25x^2-8$' (see link E). The link happened by their effort to imagine its graph by looking at its equation. After sketching the graph based on the symmetry between the graphs of $y=-0.25x^2$ and $y=0.25x^2$, they imagined a horizontal translation of 8 units on the graph of $y=0.25x^2$, instead of a vertical translation. When they saw their confusion, they turned their attention to 'value of y in turning point'.

Turning point was also used by John & Tanya as a way of recognising parabolic shape. Parabola for them was a 'curve with a turning point'. The evidence of that was the way they called the graph of $y=7\sin(0.25\pi x)$: 'many parabolas'. Link D shows that this idea agrees with John's perception of turning point in the pre-test.

In DG Parallel John & Tanya discriminated turning point in two ways. The first perception was expressed by John while analysing the striker of $y=7\sin(0.25\pi x)$: 'change of the orientation of the striker in relation to the orientation of x'. Although it corresponds to the idea 'point where the graph changes from increasing to decreasing or vice-versa', which they discriminated in FP, these perceptions were not spontaneously linked. The second perception was indicated by John while analysing the striker of $y=0.25x^2$: 'bound of the motion of the striker'. This perception was generalised by John & Tanya to the strikers of $y=0.25x^2-8$, $y=7\sin(0.125\pi x)$ and $y=-0.25x^2$. Note that, unlike in their pre-test and in FP, the idea of 'bound...' was localised in y. Despite identifying turning point as being 'bound of the motion of y', these students did not distinguish whether it was maximum or minimum. Another point considered by John & Tanya was 'value of y of the turning point' which was used to describe all the above-mentioned strikers.

Links G, H and I show that John & Tanya brought to DG Cartesian arguments that they used in DG Parallel to localise turning point: 'bound of the motion of y', 'change of orientation' and 'value of y'. Link F presents John & Tanya's link between the ideas of turning point as 'change of orientation...' and as 'point where a graph changes from increasing to decreasing'. The evidence of this link was that they usually waited for the change of orientation in the striker to identify the shape of its graph.

It is interesting that John & Tanya presented two corresponding ideas which were not linked: 'top of a curve' and 'bound of the motion of y'. Both of these perceptions attributed to turning point a perception of boundary. Therefore, the idea of turning point as 'bound of the motion of y' seems to represent an isolated perception articulated in DG microworlds. It is also interesting that in DG microworlds the perceptions developed by these students are closely related to motion.

Links J* and D present John & Tanya's link between turning point as being 'point where a striker changes orientation' to their pictorial perception of turning point from the pre-test. This connection happened while they were matching graphs to strikers. The students awaited the return of the striker to decide if it represented a parabola.

As shown by link K* John & Tanya recognised that 'bound of y' in the Cartesian system corresponded to 'bound of the motion of y' in the strikers. This link happened when they were answering direct questions about the perceptions of turning point that they constructed in DG microworlds.

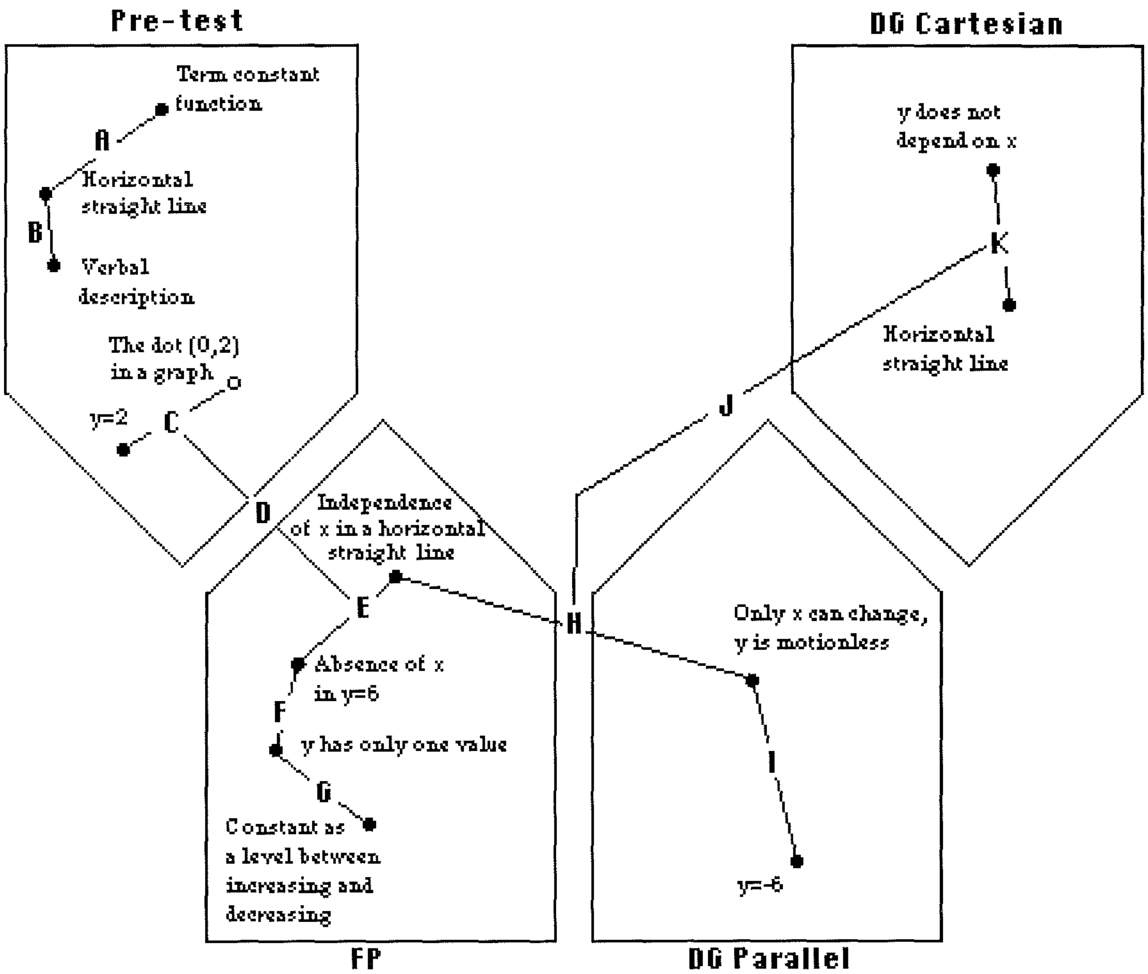
3.2 Constant function

In the pre-test, John & Tanya identified constant function by its term and graphic representation and verbal description. Link A represents Tanya's connection between the term 'constant function' and 'horizontal straight line'. Link B shows that both students were able to trace the graph of constant function from a verbal description — a stopped car — as a horizontal straight line.

The students did not match equation to graph of a constant function. Tanya plotted the graph of y=2 as a dot in (0,2) (see link C) while looking at the equation. By starting working with FP the students were motivated to continue exploring the connection between the algebraic and graphic representations of a constant function. The interaction with FP gave the students the opportunity to revise link C. Tanya was trying to check her prediction of the graph of y=6 as a dot at (0,6) (see link D) when she traced it in FP. On trying to make sense of the graph at the screen, she linked 'the absence of x' in the equation to 'the independence of x' in the horizontal straight line (see link E). She affirmed "it is a straight line because x can be any value, but y will always be 6".

Diagram 3.2

John & Tanya's perceptions of constant function



In FP, John & Tanya also connected 'y has just one value' and the fact that 'the graph of y=6 does not increase or decrease' (see links F and G). Firstly, John characterised the graph of y=6 as a 'level between increasing and decreasing' to distinguish it from the graph of y=2x. Secondly, Tanya previously distinguished these graphs by their range: 'y has just one value' for the graph of y=6 and 'y has many values' for the graph of y=2x. Therefore, her argument was the same as John's perception. She explained the similarity arguing: if "y=6 has just one value, it has no variation".

Since the starting activity with DG Parallel John & Tanya characterised the strikers of y=-3 and y=6 as being motionless. This characteristic was used by John to group these two strikers together. A second point used by this pair to characterise these strikers corresponded to the idea of 'y is independent of x' which was added in the description of both strikers: 'only x can change but y is motionless'.

Link H shows that John & Tanya connected 'the motionless behaviour of the strikers' of y=6 and y=-3 to 'the constancy of y in their graphs'. On describing the striker of y=6, Tanya identified the fact of 'y is motionless' as a cause of the horizontal straight line shape of its graph. As John & Tanya had already constructed the perception 'y did not vary' to 'horizontal straight line' in FP, they easily matched the graph of this striker.

Tanya & John also used 'only x can move, y is motionless' to build up a corresponding equation (see link I). This was reached by successive connections between equation and strikers. John was trying to find out the striker of y=x-6 through Tanya's description. He argued that the striker was -6 when x was zero and then its equation was y=-6. As soon as he said that, he imagined the equation y=-6 in DG Parallel noticing that this equation should correspond to a motionless striker.

In DG Cartesian John & Tanya just confirmed link H between 'y does not depend on x' and the shape of its graph (see links J and H). Tanya left it very clear when analysing the striker of y=-3. She said that it was "a straight line with straight angle" and "the triangle [x] moves, moves, but y does not move".

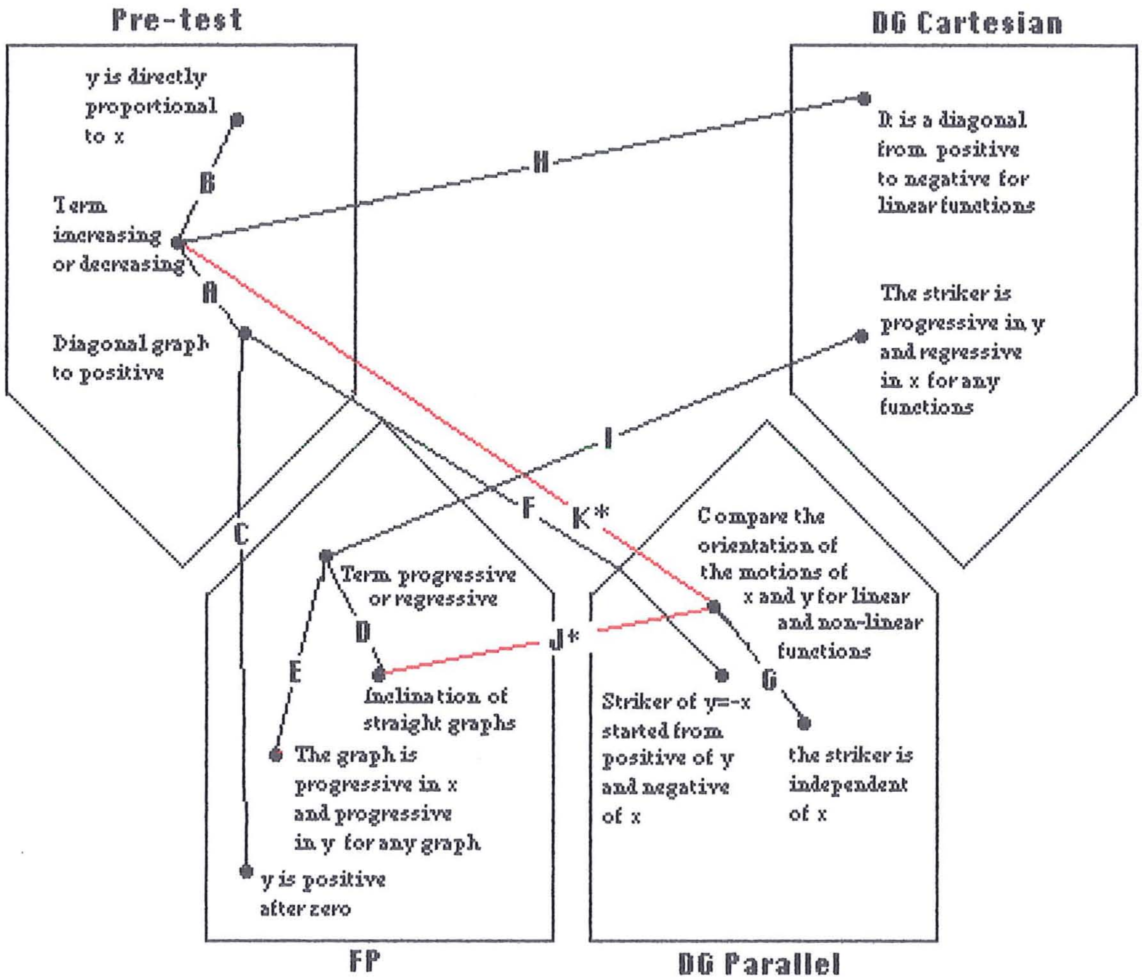
Diagram 3.2 shows that by connecting the perceptions of constant function in the same kind of function through different microworlds, John & Tanya constructed a variational perception of horizontal straight line which was linked to 'y is independent of x'. The diagram also shows that this pair of students connected their perceptions throughout the research environment.

3.3 Monotonicity

In the pre-test John & Tanya perceived monotonicity as a property restricted to linear functions which was generalised to other families of functions by polarised rule. For instance, John defined 'increasing function' as being "a function whose values are moving in diagonal [direction] to positive orientation" (see link A). According to Tanya, it was "a function in which y is directly proportional to x" (see link B). Note that these perceptions are valid only for linear function, not for hyperbolic functions. In the hyperboles John used the rule he created for identifying the property.

Diagram 3.3

John & Tanya's perceptions of monotonicity



In FP John & Tanya developed a variational analysis of this property by analysing which variable is increasing or decreasing using the terms 'progressive' and 'regressive'. This perception was completely separated from the previous idea of increasing function. It is interesting that on talking about monotonicity for linear functions, John used a pictorial perception. He recognised it by 'direction of the straight line' associated to the rule "it [y] is positive after [x is] zero" (see link C). On the other hand, by comparing the graphs of y=7sin(0.25 x) and y=x, John constructed a generalisable and variational perception of monotonicity — 'one [y=x] is always progressive, the other [y=7sin(0.25 x)] changes'. This perception was generalised by John to the parabolas while investigating this idea using a horizontal stretch between the graphs of $y=-0.25x^2$ to $y=-0.25(x/6.707)^2$.

Their next step in the development of the idea of increasing as 'progressive' was the separation of the behaviour of x and y. By trying to generalise 'progressive' to the graph of y=6, they started to analyse what was happening to x and to y. They said that the constant function is progressive only in the x-axis but it is not progressive or regressive in the y-axis. Later, they analysed progressive and regressive in both axes for the other graphs (see link E).

The last step was their synthesis between the idea of monotonicity as 'progressive' and 'angle that a straight line forms with the x-axis' for linear functions. By investigating the idea of 'progressive' with a horizontal translation in the graph of y=x, John explained that the characteristic of being 'progressive' did not change. Later, the students explained that up to 90 degrees straight lines stay 'progressive' (see link D). Moreover, Tanya explained that in graphs with curvature they cannot see angle. That is why the above link was restricted to straight lines.

In DG Parallel 'orientation of the motion of a striker' was an important aspect used by the students to characterise the strikers. As a starting point, the absence of control in a first exploration of DG Parallel was interpreted by John & Tanya as 'y is independent of x'. The constant oscillation between 'y follows x' and 'y does not follow x' of the striker given by $y=7\sin(0.125\pi x)$ encouraged John & Tanya to think that 'this striker was independent of x' (see link G). Later, the idea of monotonicity was discriminated and generalised by John & Tanya in strikers of DG Parallel as 'y follows x'.

The idea 'orientation of the motion of the striker' was constructed by John & Tanya by many analyses and comparisons of the strikers. Firstly, Tanya used this idea to characterise the striker given by y=x. She associated three different aspects in her characterisation: 'x is equal to y', 'x and y both move to the same side' and 'x has same speed as y'. Secondly, by arguing if the striker of y=2x could correspond to the description 'y follows x' and by analysing the idea in the striker of y=x-6, Tanya realised the different aspects involved in her idea of 'y follows x'. Therefore, both students moved from these associations using 'y follows x' only for the idea of 'x and y both move to the same side'. It is important to remember that this characteristic was not linked to the term 'increasing' from pre-test. At last, by overcoming the limits of these associations, the students generalised this perception of monotonicity to the strikers of $y=0.5x^2$ and $y=7\sin(0.25\pi x)$ which are non-linear functions. In the first striker, Tanya identified the domain where 'the striker follows x' from the domain where 'the striker does not follow x'. In the second striker, they just identified that "sometimes y follows x, sometimes it doesn't".

Link F shows that John & Tanya only used the term 'decreasing' after matching the graph and the striker of y=-x. It is interesting that they did not link it to 'y does not follow x'. Instead, Tanya associated it to 'it started from positive of y and negative of x and it finished in negative of y and positive of x'.

As diagram 3.3 shows, the terms 'increasing' and 'decreasing', which they learnt at school, were linked by John & Tanya to 'inclination of straigt line and to rules which were created using positive and negative values. It seems that the use of the term 'increasing' represents a didactical obstacle to their link to a variational meaning of monotonicity.

Owing to the presence of shape in DG Cartesian representation, the above pictorial way of discriminating monotonicity appeared stronger in John & Tanya's work. While working with the striker of y=x, Tanya exclaimed "it is a diagonal from negative to positive for both [x and y]". In the same way, John characterised the striker of y=-x as being 'decreasing' (see link H).

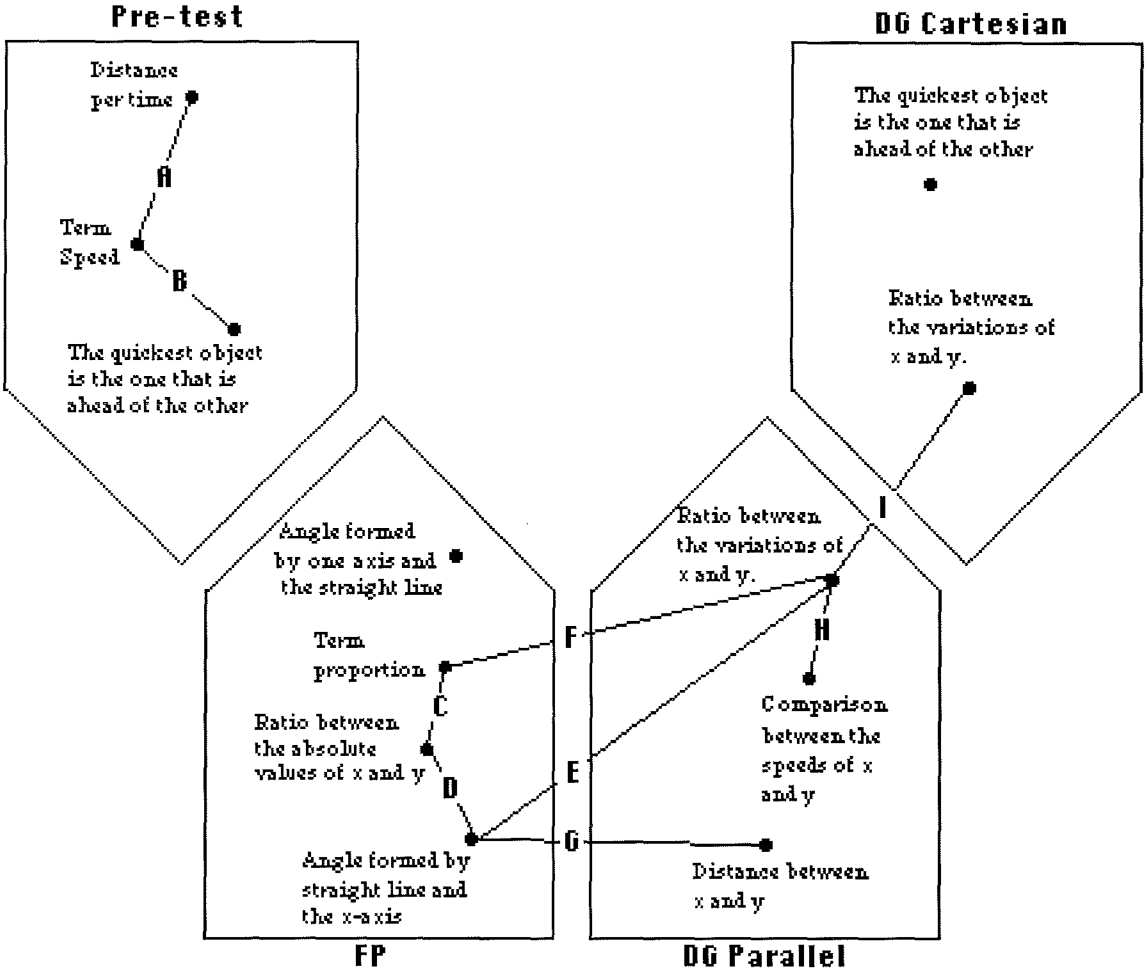
On the other hand, a variational perception of monotonicity was brought from FP to DG Cartesian by John & Tanya while discussing monotonicity. After John's characterisation of the striker given by y=-x, Tanya brought the idea of 'progressive in x and regressive in y' to describe it (see link I). Despite using the term 'increasing in x and decreasing in y', Tanya considered this use as being different from the idea expressed in the term 'increasing'. Later, this idea was generalised to the strikers of y=2x, y=x-6 and $y=0.25x^2$ when considering domain of the last striker.

In the final interview John & Tanya linked 'orientation of the motion of y' from DG Parallel to the terms 'progressive' and 'regressive' through 'inclination of graphs' (see link J*). In the case of linear functions, the students also linked these perceptions to the term 'increasing' or 'decreasing' (see link K*). In the first instance, John & Tanya linked 'inclination' which they called 'increasing' to 'orientation of the motions of x and y'. Later, on analysing graphs of parabolas, they started to analyse the monotone behaviour of x and y separately in the graphs to match with 'orientation of the motion of x and y' in the strikers. Finally, on being asked about the term 'progressive' which they created in FP, John & Tanya linked this to 'y follows x' or 'y does not follow x'. Therefore, this perception, which was isolated in DG Parallel, was synthesised to the terms 'progressive' and 'regressive' in the final interview. In conclusion, the terms 'increasing' and 'decreasing' derived from school knowledge continued to be used by John & Tanya confined to linear functions.

3.4 Derivative

Diagram 3.4

John & Tanya's perceptions of derivative



In the pre-test John & Tanya knew the definition of speed (see link A) but they had difficulties in discriminating the idea of derivative as speed through different representations. For example, they did not use slope to interpret speed in graphs. As far as equation is concerned, John & Tanya had difficulties in using the formulas and did not link coefficients to derivative. Nonetheless, they used intuition to find out which object was quicker. Despite knowing the definition of speed, John & Tanya interpreted it by the positions of the objects instead of their variations. These reasons were very close to the idea that 'the quickest object must be ahead of the slowest one' (see link B).

As Diagram 3.4 shows, John & Tanya's perceptions of derivative in the research environment were different from the pre-test. The students started to consider 'variations of x and y', and also linked these perceptions of variations to inclination of straight line (see links D and E). They started considering 'absolute values of x and y' in FP. In a continuous process throughout the microworlds, John & Tanya reached the perception of derivative as 'the ratio between the variations of x and y' in DG Cartesian.

In FP, on trying to obtain the graph of y=6 from the one of y=2x, Tanya discriminated slope as being 'the angle formed by one axis and a straight line'. Nevertheless, Tanya was not able to interpret it in a functional way. The idea of derivative was discriminated in a pictorial way. Later, in a special moment in my observations, she reported this inability.

As for John, he perceived derivative by giving an order for monotonicity. While exploring monotonicity stretching vertically the graph of y=abs(x) to the one of y=2abs(x), he argued that "it [the graphs] became more increasing" to distinguish the two graphs. He also linked this 'more increasing' or 'less increasing' to the different angles the graphs form with the x-axis. However, he was not able to measure this ratio of increase.

Tanya perceived derivative as 'the ratio between absolute values of x and y' while stretching vertically the graph of y=x to the one of y=0.5x. By searching an equation for the new graph, she constructed this idea as well as linking it to the 'angle' (see links C and D). Unfortunately, she constructed the link based on the value of angles, instead of only comparing them.

The parallelism between straight lines obtained by vertical translation, while exploring the above-mentioned link, motivated Tanya to use 'the ratio between the absolute values of x and y' for graphs of affine functions. Despite building this perception of derivative and giving an example of affine function, in which their idea does not work, she did not check the value of 'the ratio' in the new graph. This was a critical moment for generalising and realising the incompatibility of these perceptions which she missed.

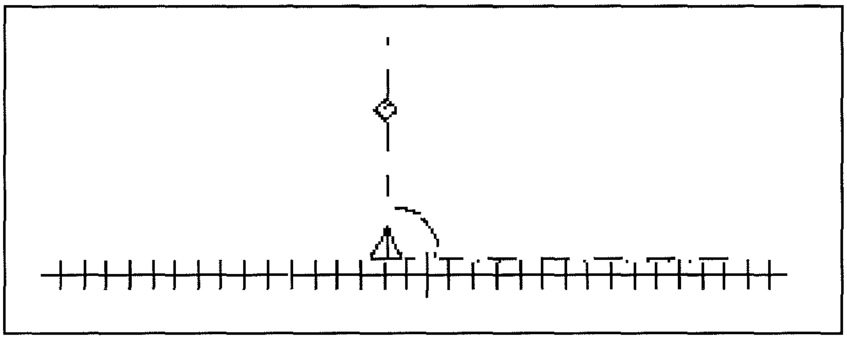
Note that by linking 'angle' to 'the ratio between the absolute values of x and y' as well as by arguing that they could not see angle in graphs with curvature, John & Tanya did not even try to calculate this ratio in graphs with curvature. Therefore, I consider that link D created a barrier to the construction of the idea of constant and variable derivative.

In the starting activity with DG Parallel, without knowing the strikers represent functions, John & Tanya discriminated derivative by comparing the speed of the strikers. They characterised speed as being slow or quick. After being informed how the strikers represent functions, John & Tanya evolved this perception by comparing the speeds of x and y. For instance, on reading the idea 'y has the same speed as x' in Tanya's characterisation of the striker of y=x, John discriminated the speed of the striker of y=2x as 'y is quicker than x' and the one of the striker of y=-x as 'y has the same speed as x'. Nonetheless, they were still limited to analysing speed in strikers of 'linear' functions.

On exploring their perception of speed in the striker of y=x-6, John brought the idea of 'the ratio between the values of x and y' from FP to DG Parallel. This connection was evident because he used the same term created by themselves in FP (see link F). At this moment, John explained "while the triangle grows one unit, it [y] also grows one unit". The perception of derivative changed to 'variations of y and x', instead of 'absolute values of x and y'.

Figure 3.1

Scheme of the imaginary angle in DG Parallel



The idea of speed became stronger in John & Tanya's characterisation of the strikers after they linked 'angle' from FP to speed from DG Parallel. Their first attempt at linking was a direct link to an imaginary angle (see figure 3.1). By linking the behaviour of the striker given by y=x to its equation, Tanya remembered that it corresponds to the straight line with 45 degrees. So, she became curious to find in DG Parallel an idea corresponding to this angle. She imagined that an angle of 45 degrees in Cartesian graph should correspond to an angle of 90 degrees in DG Parallel based on this striker. As a consequence of that, 'the distance between x and y' should be fixed (see link G). Note that she did not try an angle with same measure but an angle as an object. Link G was revised by her analysis of the striker given by y=6. She noticed that despite having null angle in graphs, the distance between x and y varied.

After classifying the strikers of y=x, y=-x, y=2x and y=x-6 as 'straight line with obtuse angle', the students tried to determine the angle of each striker. Note that, up to this point, the students had not made clear the link between angle in FP and speed in DG Parallel. By comparing the strikers with the same speed to the graph with the same angle, Tanya realised that 'angle' in the graph corresponds to 'speed' in DG Parallel. By isolating the invariants of the functions in each representation, John & Tanya established link E limited to linear functions. Even to parallel straight lines, they linked the same inclination of graphs to 'same speed of x and y' in DG Parallel, as well as to the graphs of y=-x and y=x. For these reasons, I observed that the comparison between corresponding examples in different microworlds was decisive for this link.

It is interesting that John & Tanya used speed linked to inclination while comparing two sines in order to decide which graph to match to each striker in the final interview. In fact, it seems that they were not clear about the difference between slope and curvature for this kind of graph.

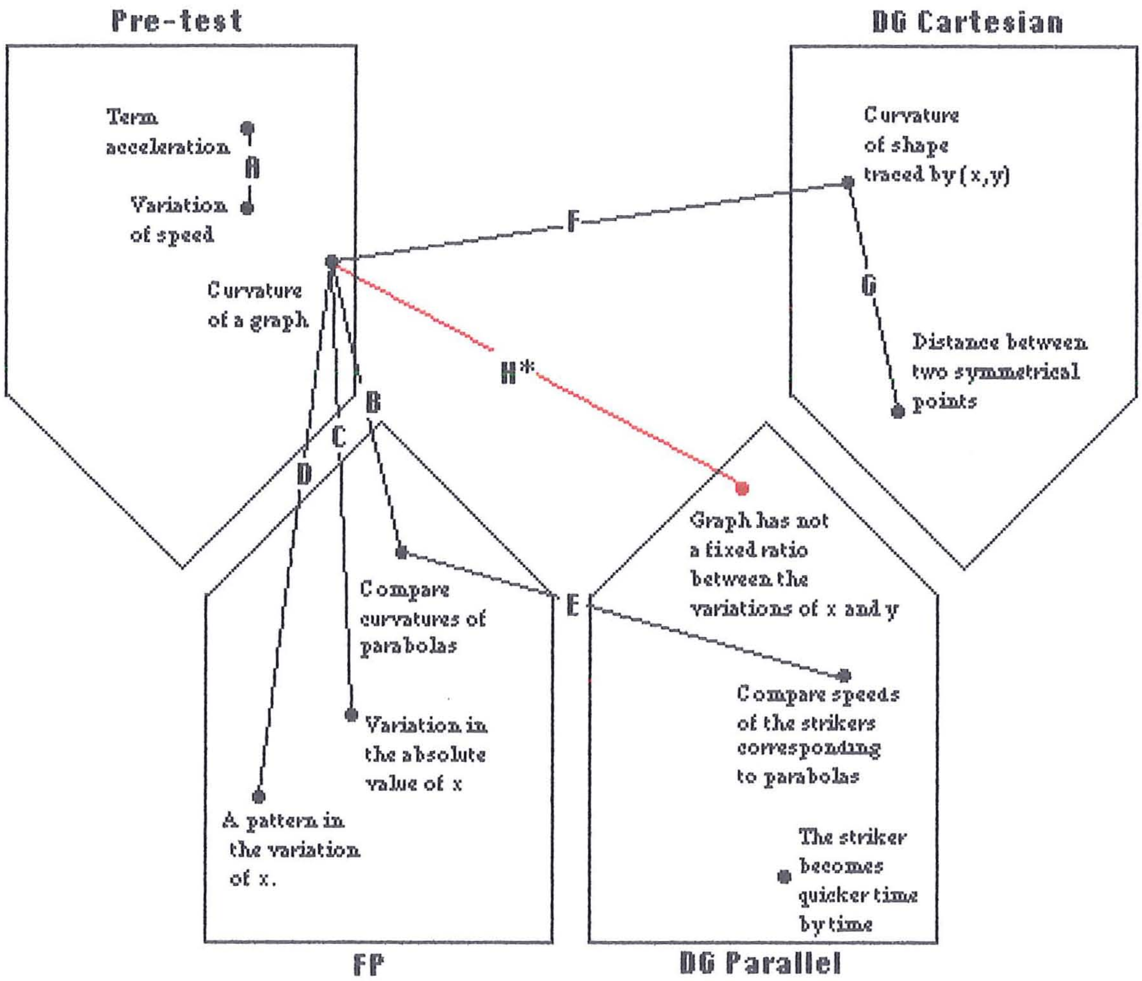
Note that both students seemed to have two different ways of discriminating derivative in DG Cartesian. As in the pre-test, in DG Cartesian Tanya associated the idea of 'bigger derivative' as 'being quicker' recognised by 'arriving first' or 'being in front of'. For instance, on comparing the striker of y=2x and y=x-6, Tanya argued that the first one was quicker than the other because it was the first to arrive. This process of building the above-mentioned perception of derivative which they called proportion inhibited their previous idea of speed which only appeared again in their work with DG Cartesian by the existence of shape. John discriminated derivative by 'the ratio between variations of x and y', as he did in DG Parallel. For example, on comparing the above-mentioned strikers he argued: "they [x and y of y=x-6] are proportional relating to motion, because it grows half unit and the triangle grows half unit... the pink striker [y of y=2x] is one in one". John generalised the idea of derivative as 'the ratio between the variations of x and y' to the strikers of $y=7\sin(0.25\pi x)$ and $y=7\sin(0.125\pi x)$. In fact, he over-generalised this idea without really verifying it.

3.5 Second Derivative

Tanya defined acceleration as 'variation of speed' in the pre-test (see link A). Despite discrimining curvature of graphs, John & Tanya did not connect it to acceleration. This lack of connection was also apparent in their sketch of a graph from a verbal description. There, they used a straight line to represent constant speed as well as variable speed. Another point observed in their pre-test regarding curvature was that both students said that two parabolas vertically translated had different curvature.

Diagram 3.5

John & Tanya's perceptions of second derivative



The interaction with transformations of graphs in FP encouraged the students to search for a functional correspondence of curvature concluding with a 'pattern in the variation of x' for parabolas with same curvature (see links B and D). Table AIV-5.2 shows the evolution of their perception of curvature while they were exploring translations and stretches. This perception was first based on absolute value of x and y, instead of a variation of them (see link C). That is why John became confused when Tanya tried to localise and to compare the value of x in the graphs of $y=0.25x^2-8$ and $y=-0.25x^2$ (see link B). Then, they associated curvature with 'variation of x' for a fixed y. This comparison gave to the students a critical moment to revise this association. After the above mentioned passage, John insisted on investigating the idea of curvature as 'variation of x' for parabolas with the same curvature. While translating the graph of $y=0.25x^2-8$ to the one of $y=0.25x^2$, he noticed a pattern of variation but without the same value of y (see link D). Then, he affirmed "despite changing the value of y, they will be proportional".

In DG Parallel, on trying to match the graphs and the strikers, John & Tanya linked curvature to speed, instead of acceleration (see link E). In fact, this provided a parallel with their link of speed and angle for linear functions. This parallel was motivated by their need to distinguish the strikers of $y=0.5x^2$ and $y=0.25x^2$. They concluded that a graph with a "more bent curvature will [correspond to a striker that] moves quicker". The same idea was generalised to see speed and curvature in strikers of sines.

Note that the idea of second derivative as 'variation of speed' was only mentioned informally in the starting activity with DG Parallel, before they needed to describe the strikers. John noticed the variation of speed in the striker of $y=0.25x^2-8$ and mentioned that "it is becoming quicker time by time". This idea was not explored until the final interview. Another indication was that they did not use the acceleration to decide whether the striker of $y=0.25x^2-8$ was a straight line or curve. So I concluded that they did not spontaneously link the idea of the striker becoming quicker and quicker with the curvature of a parabola.

Diagram 3.5 suggests that the idea of second derivative was used by John & Tanya only in a pictorial way in microworlds which contain the Cartesian representation. On the one hand, the shape of the graph in DG Cartesian motivated them to bring the idea of curvature from their previous knowledge to characterise the strikers of parabolas (see link F). On the other hand, absence of a shape traced in the screen of DG Cartesian promoted in the students a curiosity to try a functional correspondence to measure these curvatures. At this time, Tanya demonstrated how they measured the curvature of a parabola. After identifying the strikers of $y=0.5x^2$ and $y=0.25x^2-8$ as corresponding to parabolas, Tanya argued that "the first striker was narrower than the second one" meaning that its curvature was more curved than the other. She observed the 'distance between two symmetrical points' (see link G). The method is compatible to the term she used and it also agrees with the results of their pre-test.

While classifying the striker in DG Cartesian, the students failed to take advantage of a critical moment in revising this method of measuring curvature. After matching the graphs of parabolas to the strikers, they used different justifications to distinguish curvatures by the strikers. They used the method mentioned in the last paragraph to distinguish the curvature of the striker given by $y=0.5x^2$ from the others. In contrast, on trying to explain why the strikers of $y=0.25x^2$, $y=-0.25x^2$ and $y=0.25x^2-8$ had same curvature, Tanya did not apply the same rule. Instead, she created a new rule "for each x, the point [sprite of (x,y)] was over the triangle". Note that she was trying to justify something she already knew from the shape of Cartesian graph. Therefore, Tanya did not notice that the last rule was valid for any striker, in particular to the striker of $y=0.5x^2$.

In the final interview, by direct questions, John & Tanya reached the link between being curved and 'not having a fixed ratio between variations of x and y' (see link H*) — an idea presented in diagram 3.4. Nonetheless, this link was not straightforward. They followed the same path as Bernard & Charles (see section 2.5), which depended on the close relation between curvature and existence of turning point.

3.6 Range

In the pre-test John & Tanya discriminated range in two different ways: as 'length of interval that y can reach' by John (see link A), and as 'value of y that graph can reach' by Tanya (see link B).

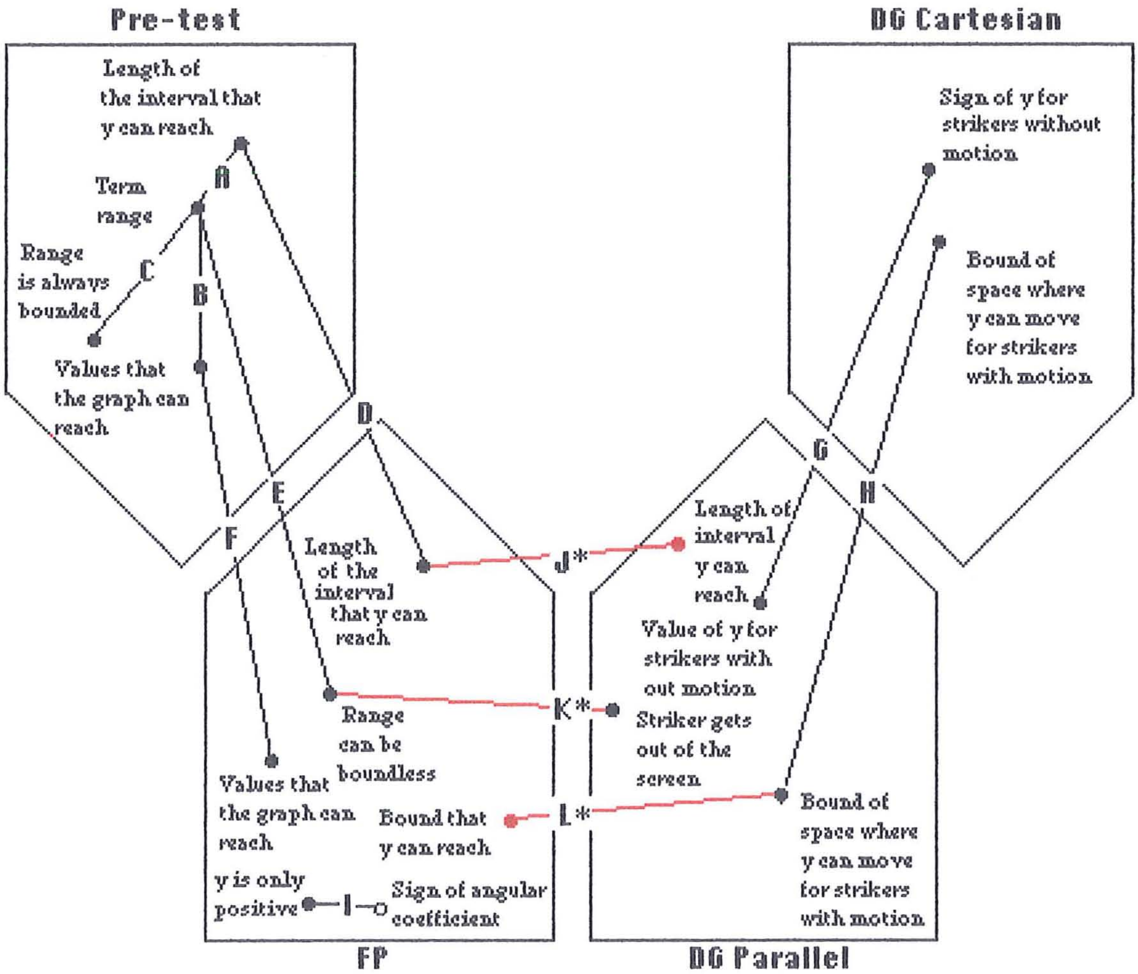
The interaction with FP led these students to discuss the meaning of the term range. They discussed whether range was the amplitude or 'points where y can reach'. John observed both perceptions by altering the range and its amplitude in the graph of $y=7\sin(0.25\pi x)$ using vertical translation and vertical stretch, respectively. He observed "in the other command [vertical translation]... the extension of the range doesn't... doesn't... it didn't change. What did change was the position of the range. This one [vertical stretch], it modifies the extension of the range..." (see links F and D). Despite noticing the difference between the two perceptions, John adopted amplitude as the meaning of the term.

Three points were crucial for the students to realise the difference between the two different ideas: generalising their perception of range to the parabolas of $y=0.25x^2$, $y=-0.25x^2$, and $y=0.25x^2-8$; investigating range while translating the graph of $y=0.25x^2-8$ vertically when Tanya argued that it was changing while John affirmed that it continued being infinity; revising the interpretion of a 'graph as being limited to the screen' which they presented in their pre-test (see links C and E). The use of FP encouraged the students to extrapolate range out of the screen.

From the last discussion a cognitive obstacle rose expressed by John as "anything that is null doesn't exist, does it? The only null thing that exists is the number zero". On exploring range stretching the graph of $y=x$ vertically, John argued that the range of any graph of y=ax was infinity and the only way to change the range was to place the graph as y=0. Nonetheless, for him anything with dimension zero could not exist. Tanya disagreed with his arguments saying that range was zero because it is "the part of the y [-axis] that a graph can reach". At the end of the activities in FP, both students agreed that the values of y in the graphs of y=-3 and y=6 change, but they were similar because they had only one value.

Diagram 3.6

John & Tanya's perceptions of range



After extrapolating from the graphic screen, John & Tanya used the idea of infinity to distinguish graphs of parabolas from graphs of linear function. John argued that straight lines "go to positive and negative infinity" but parabolas "only go to one side — positive or negative".

Another interesting association appeared when John & Tanya were describing the graph of $y=0.25x^2$ (see link I). They associated 'y is only positive' to 'positive angular coefficient'. The interesting point of this association is that it was arrived at because both were positive. Nonetheless, while translating the graph of $y=0.25x^2$ vertically, they realised that in fact the link should be between 'sign of angular coefficient' and 'positive or negative curvature'.

As diagram 3.6 shows, John & Tanya presented in their perception of range considerations which involve limit of range in all the microworlds. Nonetheless, in both DG Parallel and DG Cartesian this perception stayed completely isolated from their perceptions in the pre-test and FP.

In DG Parallel John & Tanya explored two different perceptions of range, which depended on existence of motion. That is, the students characterised the strikers without motion by the position where they stay. For strikers with motion, they used an approach involving limit. It is important that in DG Parallel this approach replaced any polarised characterisation — positive and negative — even for the striker of $y=0.5x^2$ in John & Tanya's work (see table AIV-6.2). Later, they also generalised this perception of range to the strikers of $y=2x$, $y=0.25x^2-8$ and $y=7\sin(0.125\pi x)$. Note that John & Tanya's perception of bounded range was related to the idea of motion. For example, they did not observe the strikers of constant functions as being bounded. As in DG Parallel, in DG Cartesian John & Tanya used motion of strikers to discriminate range (see links G and H).

Owing the shape of the graph — traced by (x,y) in DG Cartesian — John & Tanya generalised the idea of infinity among strikers in which 'y does not disappear from the screen before x does'. Moreover, this idea also referred to range as well as to domain. For example, while analysing the strikers of $y=6$ and $y=-3$, Tanya argued that "they were infinity in x but do not move in y". The extrapolation of the idea of boundless range of a graph was linked by John & Tanya to 'striker gets out of the screen' in the final interview (see link K*).

In the final interview as soon as I asked them to correspond range in graphs to strikers, John & Tanya pointed out that 'limit that y can reach' in graph corresponds to 'bound of motion of y' in strikers (see link L*). Moreover, after a vertical translation in a graph of sine, these students identified where the new striker can move.

In the same way, they were able to connect 'amplitude of a sine' in a graph to 'length of interval that striker can move' (see link J*). Nonetheless, this link was done after comparing two graphs of stretched sines to their corresponding strikers.

3.7 Symmetry

Diagram 3.7

John & Tanya's perceptions of symmetry

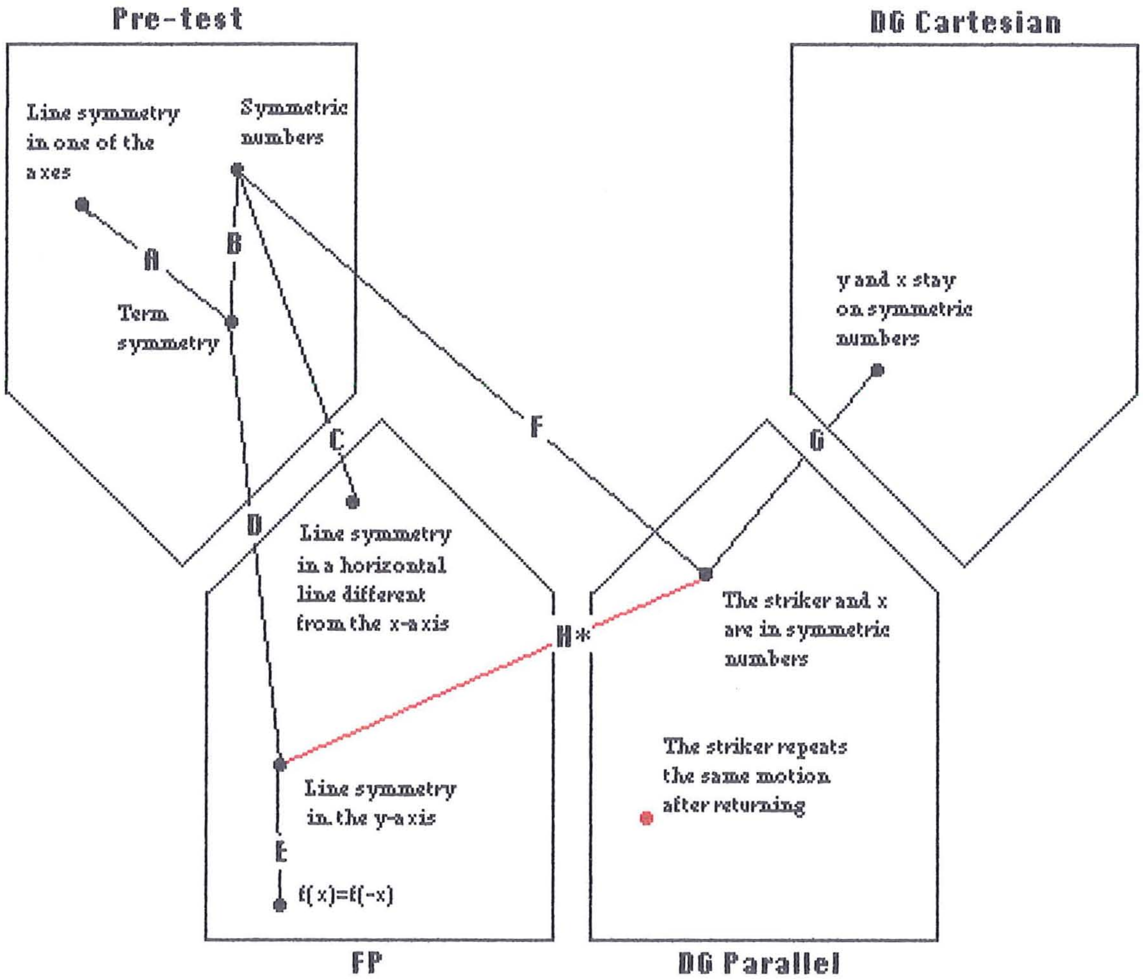


Diagram 3.7 shows that in the pre-test John & Tanya's perceptions of symmetry were limited to symmetric graphs with line of symmetry in one of the axes. For example, they did not consider a graph of sine as being symmetric. In FP, while trying to make sense of unexpected results obtained from reflection of graphs, the students realised two different points about symmetry. Firstly, they discriminated line symmetry about a line different from the axes. On trying to obtain the graph of y=6 from the graph of y=0 by vertical reflection, Tanya generalised line symmetry about a horizontal line different from the x-axis relating it to symmetric numbers (see links C). She also asserted that the only symmetry they had studied was symmetric numbers. Secondly, the students linked the invariance of the horizontal reflection in a parabola to the vertical line symmetry of the same parabola. While

reflecting the graph of $y=-0.25x^2$ horizontally, John started investigating the reason for the invariance concluding with a pictorial perception: "As it [the graph] has two equal sides, it does not alter".

The interaction with FP also encouraged John & Tanya to search for a pointwise correspondence for line symmetry in the y-axis for parabolas. Links D and E show the pointwise perception that John & Tanya reached in FP. On describing the graph of $y=-0.25x^2$, Tanya explained the symmetry by: "Oh, the point y, at the beginning it goes to a number [x], later the same point y [value] with the symmetric number in x. For example... -15 with -20, later 15 with -20". Despite developing the sense of line symmetry relating the values of x and y, this sense was restricted to line symmetry in the y-axis, which can be correlated with symmetric numbers.

John & Tanya discriminated line symmetry only in the pre-test and in FP. It seemed to be a pictorial perception which was not spontaneously perceived in DG microworlds. Link F shows that in DG Parallel they only discriminated symmetry related to symmetric numbers (see table AIV-7.2). For example, while exploring the striker given by $y=7\sin(0.25\pi x)$, Tanya used the same perception to argue that the striker was alternating from positive to negative. She added "a number and its symmetric".

In DG Cartesian, John & Tanya used the same perception of symmetry that they had built in DG Parallel (see link G). Symmetric values were discriminated in the striker of $y=-x$. Note that in DG Cartesian the students only used the relation between x and y to recognise symmetry.

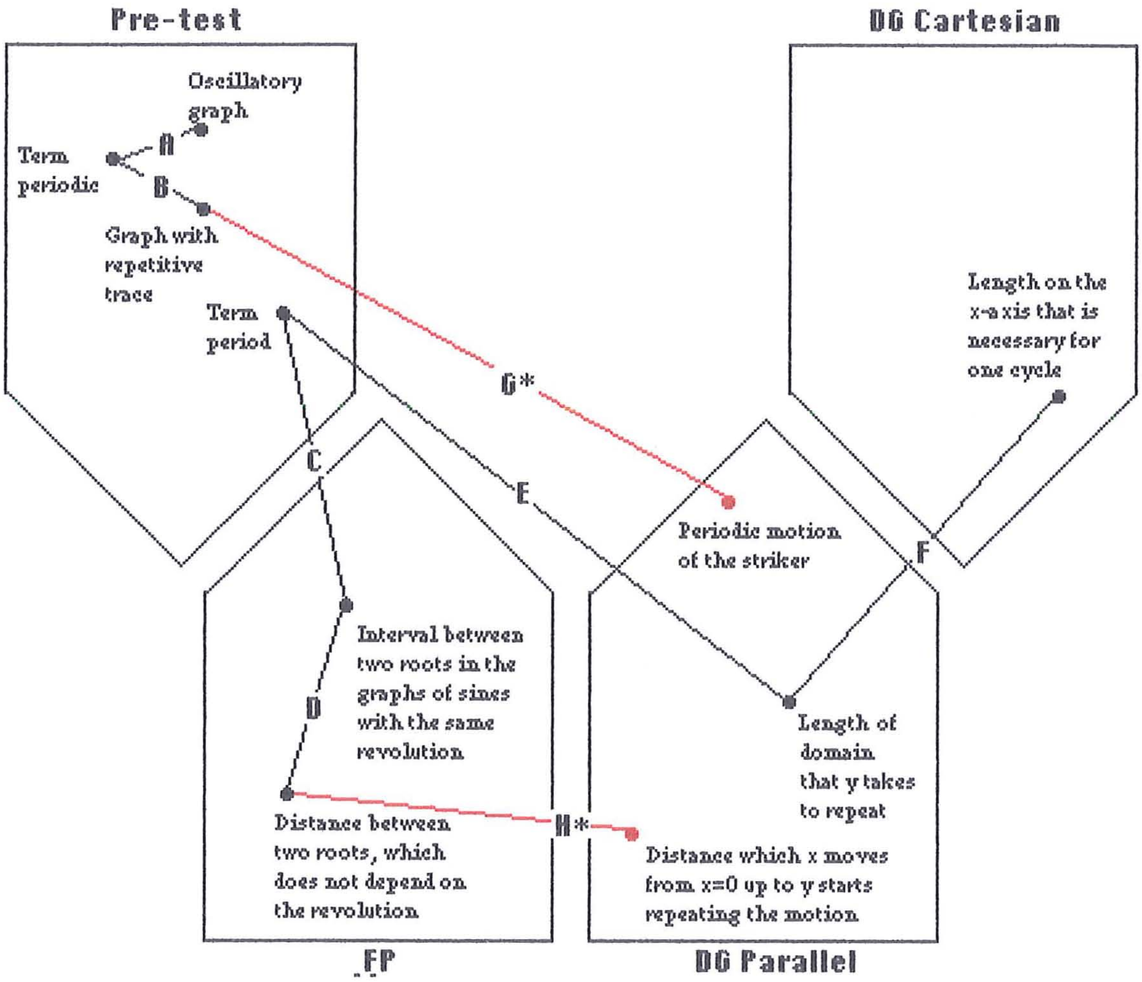
In the final interview, John discriminated line symmetry in strikers of quadratic functions variationally observing that 'the strikers repeat the same motion after returning'. Nonetheless, John & Tanya did not link this perception either to line symmetry in graphs or to the term 'symmetry'. Moreover, when asked how to identify line symmetry of parabola in strikers, Tanya again connected it to a number and its symmetric (see link H*). Moreover, she did not accept that a parabola with line of symmetry different from the y-axis was symmetric. The school emphasis on polarised knowledge represented a knowledge-obstacle which prevented the students from making this link while using the term 'symmetry'.

3.8 Periodicity

Diagram 3.8 demonstrates that John & Tanya had different perceptions of periodicity. John interpreted it as a 'graph with repetitive trace' (see link B), whereas Tanya interpreted it as any oscillatory graph (see link A). For example, she considered an oscillatory and aperiodic graph as being periodic. Despite the difference, both students considered periodicity as a graphic characteristic.

Diagram 3.8

John & Tanya's perceptions of periodicity



The meaning of period was not clear to either student. In FP since they translated the graph of y=7sin(0.25 x) vertically when trying to describe it, they brought the term 'period' to make sense of it in the graph. They at first discriminated 'period' by two roots in the graphs of sines, considering period more as the interval between the roots than as the distance between them (see link C). This perception was evident because they affirmed that the period was altering while translating the graph of y=7sin(0.25 x) horizontally. In addition, they agreed that the graph of y=7sin(0.25 x) and y=-7sin(0.25 x) had the same period because the graphs intercept the x-axis at the same points. It is important to notice the emphasis on special points, in this case x-intercept.

The exploration of FP allowed John & Tanya to discover that period of a function does not depend on cycle. They discovered it only by investigating the idea of period while stretching the graph of y=7sin(0.25πx) vertically. They generated a different graph with a different 'revolution' but with the same period. This represented a critical moment when they recognised that period was 'distance in x that a graph takes to repeat' (see link D). Unfortunately, period was only calculated as 'distance between roots'. Thus, they did not see period as invariant to another point.

It is interesting that John & Tanya's perceptions of period assumed different approaches in different microworlds. In FP it was linked to special points while in DG Parallel these students discriminated and calculated period in a functional way. The interaction with DG Parallel while discussing allowed John & Tanya to separate the ideas of period and of 'repetitive path of y'. Firstly, exploring the striker of y=7sin(0.125πx), John argued that it was a 'Roller-coaster' because y repeats. Secondly, by comparing both strikers of sines, he discriminated their period as 'length of the domain which y takes to repeat'. For example, he explained the period of the striker given by y=7sin(0.25πx) as "each 8 units x moves, it [y] makes one revolution" (see link E). The possibility to observe the representations of x and y separately in DG Parallel helped the students to explain the difference between repetitive path and period. The sequence from DG Parallel to DG Cartesian led John & Tanya's perceptions of period to separate the variables. This situation was not presented in their pre-test and in FP.

As in DG Parallel, in DG Cartesian John & Tanya discriminated the periodicity of the strikers of y=7sin(0.25πx) and y=7sin(0.125πx) by the repetitive path of y. Moreover, the meaning of period was considered by the students as the length of x necessary for one cycle (see link F). In my view, the work with different microworlds allowed the students to be clear about the difference between periodic function and its period.

It is interesting that 'periodic motion of striker' was connected to 'repetitive trace of graph' (see link G*) following a sequence of links. Firstly, the students identified that the graph corresponding to the strikers of sines should have 'many turning points'. Secondly, looking at the graphs of sines, Tanya added that the value of the turning points should be equal. Then, John concluded "they repeat... isn't it many revolutions".

Link H* shows that when asked to, John & Tanya connected their perception of period from graphs to 'distance x moves from x=0 up to y starts repeating the motion'. Nonetheless, even in DG Parallel they always fixed x at zero as a starting point.

Therefore, they did not perceive that period does not depend on the choice of the point. John also emphasised that "it [x] always starts at zero".

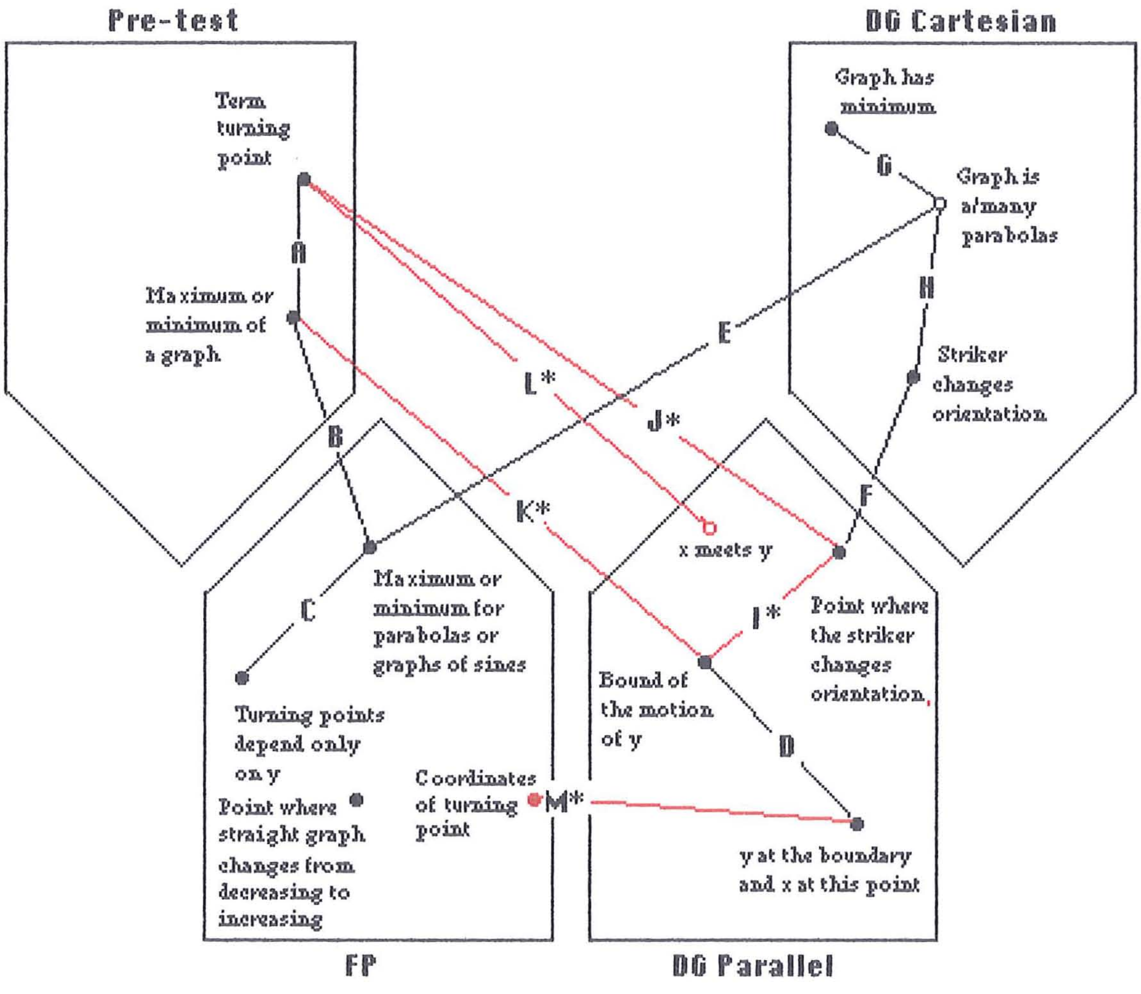
4 Diana & Gisele's perceptions of the function properties

Diana & Gisele were one of the pairs of students who followed the activities from FP to DG microworlds.

4.1 Turning point

Diagram 4.1

Diana & Gisele's perceptions of turning point



In the pre-test Diana & Gisele defined the term turning point as being 'point where a function has its maximum or minimum' (see link A). Link B shows that in FP they generalised this perception among parabolas and graphs of sines (see table AIV-1.2).

Link C confirms that Diana & Gisele perceived turning point and extreme values as having the same meaning. If the maximum of a parabola was changed, they argued that the turning point changed, otherwise, it did not. For instance, exploring the idea of turning point by a horizontal translation between the graphs of $y=0.25x^2-8$ and $y=0.25(x-17.7)^2-8$, Diana argued "their turning points are the same, just their x are different". Later, after a vertical reflection in the graph of $y=0.25x^2-8$, Diana & Gisele argued that the turning point changed.

In DG Parallel Diana & Gisele perceived turning point completely different from their perceptions in FP and in their pre-test. They identified turning point as 'the point where the striker changes orientation' for all the functions with turning points. A corresponding perception is presented in FP exclusively for graphs of absolute value function: 'point where the graph changes from decreasing to increasing'. Unfortunately, these perceptions were not linked by the students. The new perception developed by Diana & Gisele in DG Parallel enabled them to generalise it to graphs with curvature. Links E, F and H show that they used in DG Cartesian the shape of these graphs to link this perception to their previous idea of turning point as extreme values. Moreover, the perception was generalised to graphs with curvature. This process seemed to be a constructive development of ideas without barriers created by their previous knowledge.

In DG Parallel, the students also perceived turning point as 'bound of the motion of y'. Firstly, to distinguish the strikers of $y=x$ and $y=0.25x^2$, Gisele exclaimed "the other [striker of $y=0.25x^2$] is coming in opposite orientation, now, it arrives to a point where it follows [the triangle]". Then, Diana added "it doesn't go further". Table AIV-1.2 shows that Diana & Gisele generalised this perception to the strikers of $y=0.25x^2$ and $y=-0.25x^2$. Note that in DG Parallel they localised turning point as 'bound of the motion of y' without distinguishing upper from lower bound. Moreover, they did not link this perception to their previous idea of turning point as being 'point of maximum or minimum'. The perception 'bound of the motion of y' stayed isolated in DG Parallel.

The use of turning point as being 'bound of motion of y' enabled Gisele to classify the strikers by the value of their bound (see link D). She argued that there were three kinds of strikers: those which do not overtake zero, those which do not overtake -7 and 7, and those which always follow x. For this reason, I argue that they perceived the strikers corresponding to linear functions as having no turning point.

DG Cartesian was explored by Diana & Gisele as a bridge for connecting their perceptions of turning point from DG Parallel to Cartesian representation. Links F and H demonstrate that they brought their perceptions of turning point as 'point where the striker changes orientation' to recognise all the strikers of parabolas and sines as being parabolas. After recognising the shape of the graph for each striker, these students distinguished by concave or convex if it was maximum or minimum (see link G).

In the final interview Diana & Gisele linked the idea of 'bound of the motion of y' from DG Parallel to the existence of maximum or minimum in graphs (see link K*). Nonetheless, this synthesis was not straightforward. After linking 'y follows x' from DG Parallel to 'positive slope' of graphs (see links M*, N* and O* in diagram 4.3), they connected 'the point where the strikers change from y follows x to y does not follow x' to turning point of a graph (see link J* in diagram 4.1). Then, they used this connection to link 'bound of the motion of y' to extreme values in a graph. Diagram 4.1 suggests that this link was constructed through link I*. An evidence for this is their statement that a constant striker has no turning point because it has no motion.

Diana & Gisele also matched 'coordinates of the turning point' in a graph to 'value of y when it changes orientation and value of x at this time' (see link M*).

When the question was posed in opposite orientation (from FP to DG Parallel), Diana & Gisele tried to link turning point to a special point (see link L*). For instance, when asked what will happen to the striker of $y=-0.25x^2$ after a vertical translation of 10 units in its graph, they considered turning point as 'the point where x meets y' in DG Parallel. This constituted a link with special points. It is interesting that they did not observe the inconsistency between their links L* and J*.

4.2 Constant function

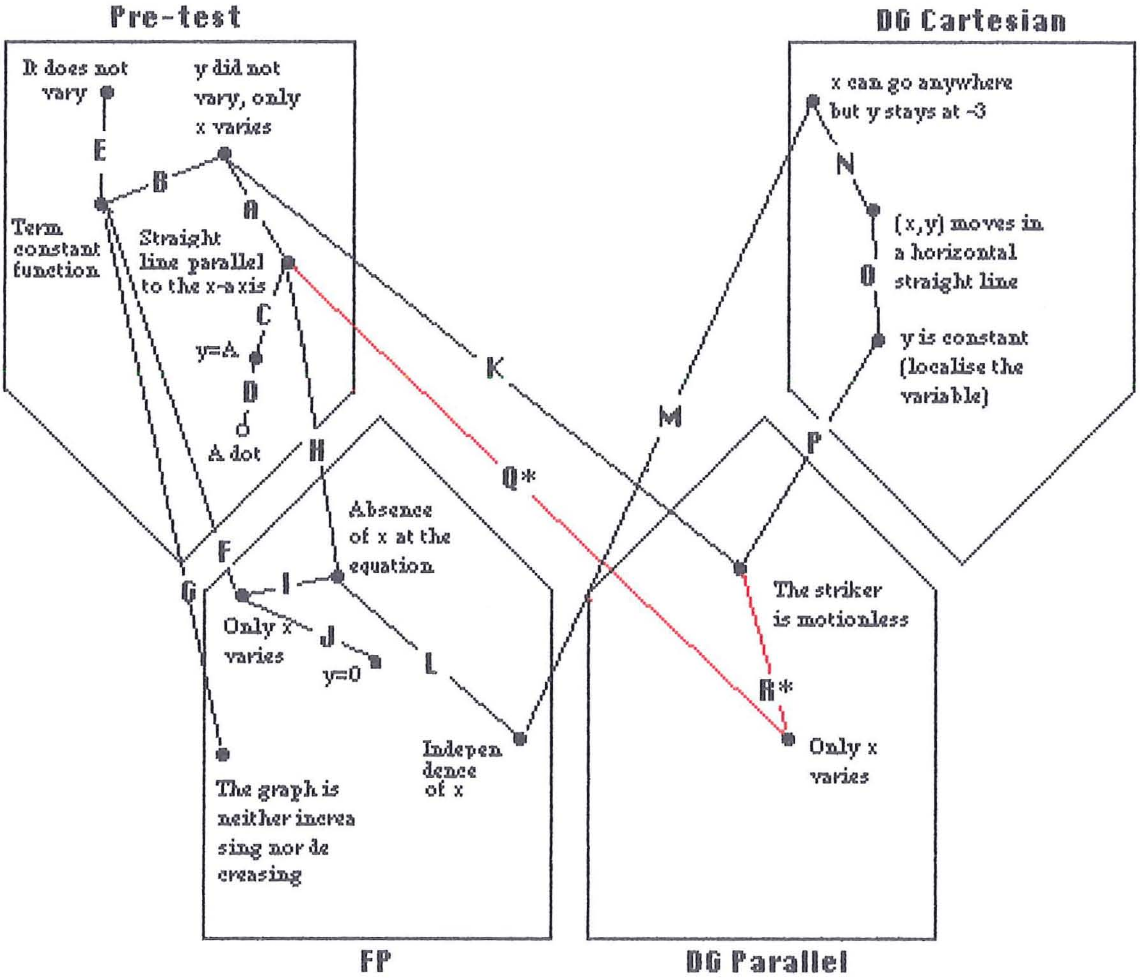
Links A, B and D are related to Diana's previous perceptions of constant function. As regards links C and E, they are related to Gisele's perceptions. Therefore, diagram 4.2 shows that Gisele & Diana were a heterogeneous pair of students regarding the previous perceptions of constant function.

Since the pre-test 'constant function' was expressed as 'y does not vary, only x varies' with recognition of its graph by Diana and as 'it does not vary' by Gisele. Only Diana localised the variable which does not vary. In FP, Gisele changed her behaviour, starting to localise the variable she was talking about. On trying to distinguish the graphs of $y=0.25x^2-8$ to $y=0$, which were obtained by a vertical stretch, Diana & Gisele discussed the meaning of the term 'constant' as being 'only x varies' (see link F). Unlike in FP where they had graphs and equations available, in

DG Cartesian they continued determining which variable is constant. While characterising the striker of y=6, for example, Gisele used 'y is constant'.

Diagram 4.2

Diana & Gisele's perceptions of constant function



Link G shows that in FP Diana & Gisele made sense of the term constant as being neither increasing nor decreasing. After trying to verify whether the graph of y=-3 was 'increasing' or 'decreasing', they argued that 'constant' means that 'it does not increase or decrease'. Note that this was the meaning given to the term, not to the graph.

After some time analysing equations and graphs of constant functions in FP, Diana connected 'absence of x at the equation' to 'y is constant' in the graph. In her pre-test she traced a graph of y=2 as the point (0,2) (see link D). In FP, she started perceiving constant in graph as 'only x varies'. Then, she linked it to 'there is no y' at the equation. Finally, on analysing the equations to find the one corresponding to the horizontal straight lines, Diana concluded link I.

I noticed a very strong tendency to associate null variation with value zero — sometimes x is zero, sometimes y is zero. Even after linking 'y does not vary' in the graph of y=6 to 'there is no x' at its equation, Diana described the graph of y=6 as 'y is equal to zero'. Moreover, Diana & Gisele mismatched 'absence of x' in equation of constant function with zero — y=0 or x=0 — when classifying the straight lines (see link J).

By the exploration of all horizontal commands in the graph of y=-3, Gisele concluded that the invariance of the graph when using these commands is due to the independence of x. Link H represents Gisele's connection between 'the straight line parallel to the x-axis' and 'absence of x' at the equation. Link L represents their conclusion which was drawn from Gisele's analysis of 'absence of x' at the equation y=-3.

In DG Parallel, the idea of constant function was discriminated by Gisele & Diana as being 'y is motionless'. Moreover, link K shows Diana's connection between 'the motionless behaviour of the striker' and the variational perception presented in their pre-test — 'Only x varies, but y is constant'. In addition, link B represents the connection Gisele made between this behaviour and the term constant from school mathematics. As in their perception of turning point (see diagram AIV-4.1), Diana & Gisele's perceptions of constant function had no direct connection from FP to DG Parallel. These students did not try to match strikers with graphs they had worked out.

The exploration of the microworlds in the sequence DG Parallel to DG Cartesian allowed Diana & Gisele to build the idea of constant function separating the behaviour of each 'object' (x, y and (x,y)) in a Cartesian representation. Therefore, these students interpreted the graph of a constant function as 'y is constant, so (x,y) moves in a horizontal straight line' (see link O). On reading 'y is constant', Diana searched for a constant striker by 'the point [(x,y)] does not move' for instance. By Gisele's remark 'it was y that was constant', Diana guessed the striker. It is interesting that Diana's expectation was also presented in their pre-test. So links O and N indicate a strength in Diana's perception of Cartesian representation of constant function. Later, by comparing the strikers of y=6 and $y=-0.25x^2$, Gisele noticed that the point follows in a horizontal straight line because y is constant (see link O). Moreover, at the end of the classification session, Diana verified that a parabola could not have constant y.

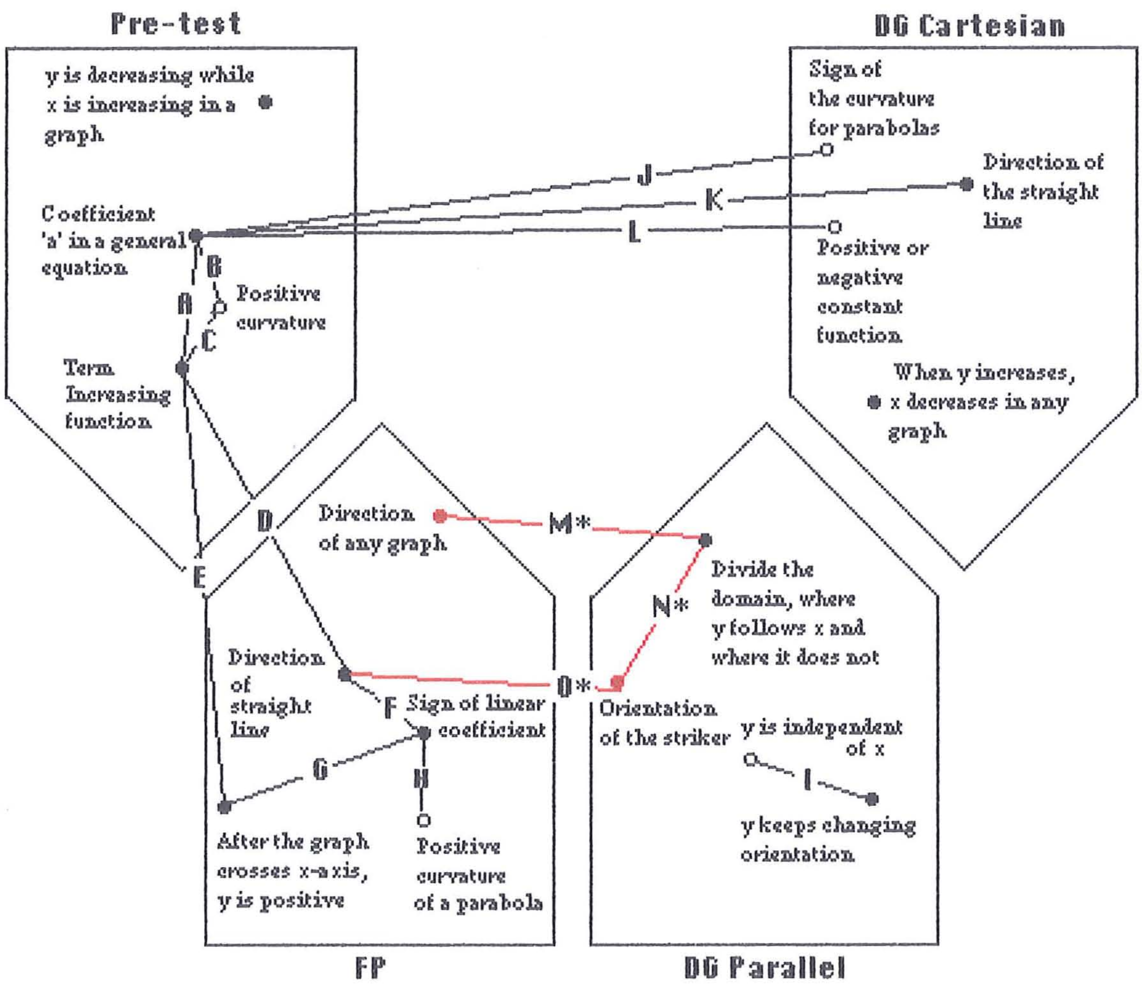
The separation of the objects — x, y, (x,y) — encouraged these students to consider a constant function of 'y is independent of x'. Link M happened while Diana was analysing the striker of y=-3. She discriminated the idea of constant by the argument 'x can go anywhere but y stays at -3'.

In the final interview the students presented links Q* and R* and a different perception in DG Parallel: 'only x varies'. This new perception was connected to 'y is motionless'. In addition, they concluded that 'only x varies' in DG Parallel representation would imply that the graph is a horizontal straight line (see link Q*).

4.3 Monotonicity

Diagram 4.3

Diana & Gisele's perceptions of monotonicity



In the pre-test, Diana & Gisele defined an increasing function by linking it to the coefficient 'a'[3] in a general equation (see link A), which caused some associations between different ideas. For example, these students classified a parabola with 'positive curvature' as being an increasing function (see links B and C), which is an association predicted in the analysis of the school curriculum (see chapter VI). As in the pre-test, in FP the students continued associating the term 'increasing' with 'positive curvature' while analysing graphs with curvature (see link H). Therefore, the term 'increasing' no longer had a sense of 'increase'. Even in the case of constant function, Diana & Gisele tried to decide whether it was increasing or decreasing function using the association.

In the pre-test, these students also developed a variational view of monotonicity in the graph of $y=3/x$, but without using the term 'increasing'. They analysed 'where y increases or decreases' in the graph.

In FP Diana & Gisele discriminated monotonicity in the graph of $y=-x$ by its direction, which was generalised to all the linear graphs excepting the one given by $y=x-6$ (see link D). Moreover, as in the pre-test, these students linked 'direction of straight line' to 'sign of the linear coefficient' (see link F).

The only attempt Diana & Gisele made to connect their previous perception of the term 'increasing' to a functional perception followed a pointwise interpretation of graphs. For instance, Gisele discriminated decreasing function in the graph of $y=-x$ by the linear coefficient at the equation explaining that 'after it [the graph] crosses x [-axis], it [y] is negative' (see links E and G). Looking back to the pre-test, this seems to be the rule used by them to decide the domain where the graph of $y=3/x$ was increasing or decreasing. Both students argued that for $x>0$ the function was increasing, when in fact, for $x>0$, y was positive.

From the starting activity with DG Parallel, monotonicity was discriminated by Diana & Gisele as 'y follows x' in the strikers of $y=-0.25x^2$ and $y=0.25x^2-8$. The students compared these strikers arguing that "up to x equal to zero, the striker [of $y=0.25x^2-8$] followed the orientation of the triangle. After that, it does not, the other striker is the opposite". The students also generalised the arguments to the striker of $y=7\sin(0.125\pi x)$, observing its oscillatory behaviour between 'y follows x' and 'y does not follow x'. Note that, unlike in FP and in the pre-test, in DG

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[3] Remember that in their mathematics textbooks the general equation presented for linear functions is $y=ax+b$ and for quadratic functions is $y=ax^2+bx+c$. There they learnt at different times that $a>0$ corresponds to an increasing linear function and that $a>0$ corresponds to a parabola with positive curvature.

Parallel monotonicity was discriminated among non-linear functions. Moreover, the change on 'orientation in motion of the striker' was what motivated the students to use this characterisation. At last, this perception of increasing was generalised among strikers of linear functions. On comparing the strikers corresponding to y=x and $y=0.25x^2$, Diana & Gisele distinguished the strikers by "any time, the striker [of y=x] follows the triangle [x]" and "up to zero, the striker [of $y=0.25x^2$] moves in opposite orientation of the triangle and after zero it follows the triangle". It is important to observe that this perception of monotonicity was not linked to the students' previous perceptions.

The students' tendency to polarise any idea into positive and negative provoked a perception of the striker of $y=7\sin(0.25\pi x)$ as 'y is independent of x' (see link I). As Diana & Gisele could not divide domains where 'this striker follows x' or where 'it does not' into positive and negative, Diana characterised this striker as 'y is independent of x' distinguishing it from the other striker of sine. Nonetheless, on describing the striker of $y=7\sin(0.25\pi x)$ without comparing it to the other striker, Diana made the same analysis as she did to $y=7\sin(0.125\pi x)$. She argued that "up to 7 the striker follows the triangle and later it starts going backwards and forwards".

In DG Cartesian Diana & Gisele discriminated monotonicity with two different perceptions without linking them: a variational perception and an association that they brought from previous knowledge. It is interesting that they almost linked the variational perception to the term 'increasing', but their previous knowledge created an obstacle to this connection. By examining the sprites of y, x and (x,y) in the striker of $y=-0.25x^2$, they started to compare the variations of x and y as "when the triangle moves from negative to positive [side], y decreases" or 'when x increases, y decreases'. While looking for the striker of y=-x described by Gisele as "when x increases, y decreases and vice-versa", Diana tried to link it to their perception of increasing presented in the pre-test. By remembering the association between graph with 'positive curvature' and the term 'increasing', she gave up trying the link. It is interesting that Diana & Gisele's variational perception of monotonicity was not limited to linear functions.

Diagram 4.3 shows that Diana & Gisele presented variational perceptions of monotonicity in all the microworlds except FP. Nonetheless, these perceptions were isolated in each microworld. In my opinion the barrier was constructed by their previous knowledge while using the term 'increasing'.

In DG Cartesian the students maintained the associations presented in FP and their pre-tests while using the term 'increasing'. While describing the striker of y=x, Diana used the term 'increasing' discriminating it by 'direction that (x,y) moves'. As they started to recognise the shapes, Diana & Gisele used the term 'increasing' to characterise the strikers of $y=0.25x^2-8$, $y=x-6$, $y=0.5x^2$. They used 'positive curvature' to mean increasing in parabolas (see link J) and 'direction of the straight line' to mean increasing in the striker of $y=x-6$ (see link K). In addition, during the task of classification of the strikers, they applied the term 'increasing' to constant functions meaning that it was positive (see link L). The meaning of this term depends on the family of the function.

In the final interview, Diana & Gisele linked 'y does not follow x' from DG Parallel to 'direction of a graph' (see link O*). Nonetheless, this synthesis was not straightforward. Firstly, Diana & Gisele discovered the equation of the striker given by $y=x$. Secondly, they matched the striker and the graphs of $y=2x$, $y=x-6$. Thirdly, on trying to match the striker of $y=-x$, they became curious as to whether the graph would be increasing or decreasing. By comparing the graphs and the strikers, they concluded that 'y does not follow x' should correspond to 'negative slope' of graph.

In a different direction of questions, Diana & Gisele were able to verify that when a graph had 'negative slope', 'y does not follow x', which enabled them to generalise this link to graphs of sines and parabolas (see links N* and M*). However, they did not link these ideas to the term 'increasing' or 'decreasing' from school knowledge.

4.4 Derivative

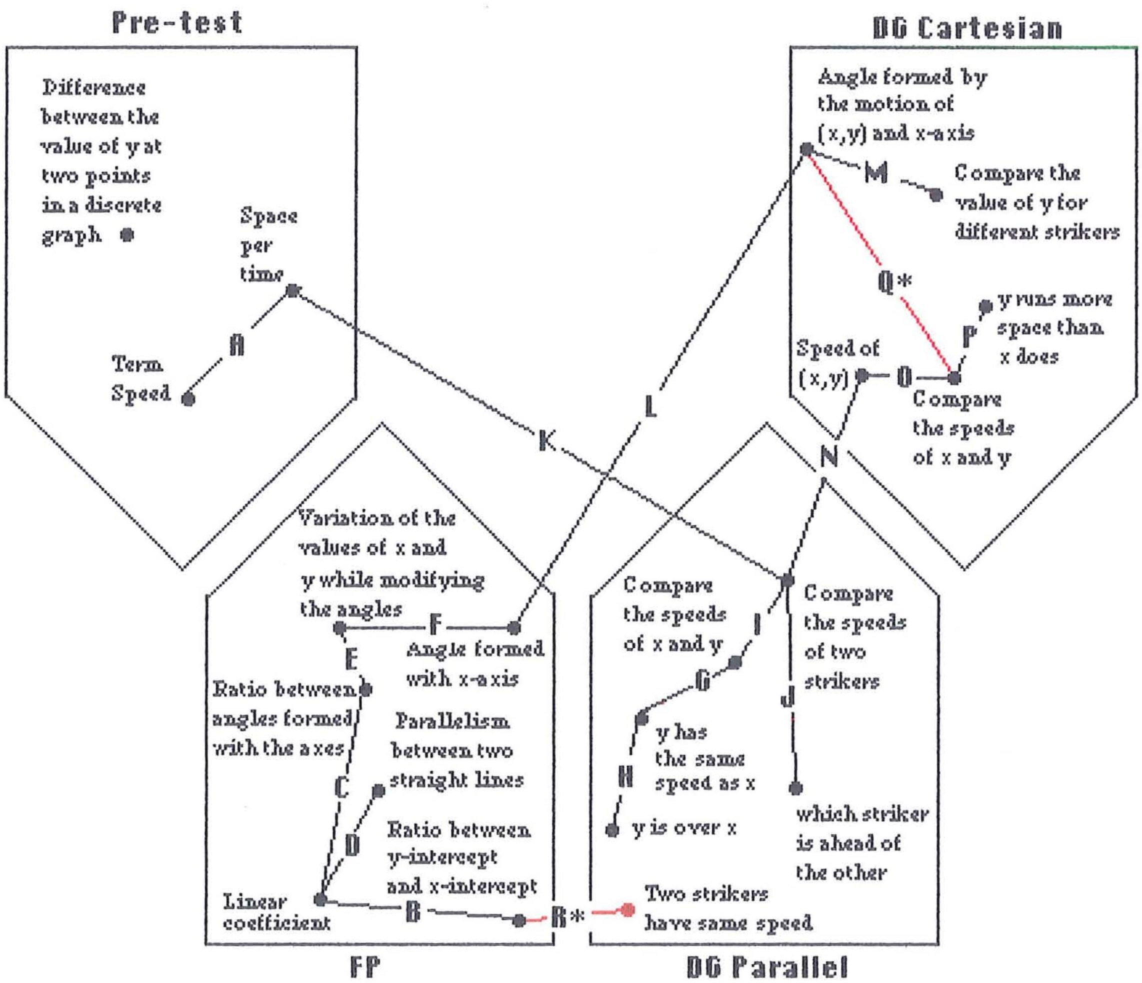
Diana & Gisele's perceptions of derivative as slope in a graph first appeared in FP. In the pre-test they interpreted derivative only in a discrete graph by subtracting the values of y. In FP Diana & Gisele discriminated slope of graph by the angle that a straight line forms with one of the axes (see table AIV-4.2). Moreover, they tried to link 'the coefficient 2 of the equation $y=2x$' to 'the ratio between the angles with the x-axis and with the y-axis' (see link C). For instance, on describing this graph, Diana wrote "the angle between its graph and the x-axis is twice the angle between this graph and the y-axis". The perception of derivative as slope was also generalised by Diana to affine functions while translating the graph of $y=x-6$ vertically (see link D).

The use of vertical translation modifying the graph of $y=2x$ encouraged Diana & Gisele to seek a functional meaning for direction of a graph. Diana verified that 'the ratio between y-intercept and x-intercept' stays invariant (see link B).

On perceiving derivative as the angle a straight line forms with the axes, Diana & Gisele limited their perceptions of derivative to linear functions. They developed a pointwise corresponding perception by considering the absolute value of y, instead of its variation. Unfortunately, in this effort they did not analyse x and y in relation to each other. They did not fix one variable to analyse the other one. For example, their analysis of the angle was 'as smaller is the angle [between the graph and the x-axis], x is bigger and y is smaller'. Another evidence of that was Gisele's arguments while stretching the graph of y=2x vertically. She observed 'x is increasing, it is becoming bigger. It is staying closer to x [-axis]' (see link E). Meanwhile, the perceptions developed by Diana & Gisele in DG Parallel considered the variations of y and x. They also fitted with the concept they learned at school: the speed (see link K).

Diagram 4.4

Diana & Gisele's perceptions of derivative



Because of the absence of Cartesian representation in DG Parallel, Diana & Gisele discriminated derivative by comparing the speeds of x and y. During the development of their perception of speed, these students passed through associations. Firstly, they discriminated the speed of the striker given by y=-x as "being almost the same as the triangle [x]". Later, Diana also discriminated the speed of the striker given by $y=x$ as having the same speed as x. Nonetheless, this perception was not a generalisation of the same speed of the striker of $y=-x$, described by Gisele. Diana associated the ideas 'y has the same speed as x' to 'y is over x' (see link H). By arguing that the striker of $y=x-6$ had the same speed as x, Gisele generated a critical moment that allowed Diana to realise and revise her association between 'y has same speed as x' and 'y is over x'. Secondly they strengthened this perception to 'y is quicker, slower and the same speed as x' to characterise the speed of $y=x-6$ and $y=2x$ (see link G). Finally, they generalised to strikers of parabolas. In DG Parallel, unlike Cartesian systems, this perception of derivative was not limited to linear functions. Diana & Gisele argued that the strikers of $y=0.5x^2$ and $y=2x$ were similar because they were quicker than x.

Another association made by Diana & Gisele during the development of the perception of derivative as speed was between the ideas 'A moves quicker than B' to 'A is in front of B'. On comparing the strikers of $y=0.25x^2$ and $y=0.5x^2$, they observed that the second striker was quicker to disappear from the screen when going from zero to the positive. However, when returning to the screen, the striker was considered to be slower than the other because it came behind the striker of $y=0.25x^2$ (see link J). When Diana & Gisele compared the speed of the strikers of $y=x-6$ and $y=2x$, they noticed that there was a difference between the ideas of 'A is in front of B' and 'A is quicker than B' (see link I).

Speed was also used by Diana & Gisele to compare strikers of parabolas. For instance, Diana & Gisele compared the strikers of $y=0.25x^2-8$ to $y=0.25x^2$ and $y=x$ arguing that in respect of speed, the strikers given by $y=0.25x^2-8$ and $y=0.25x^2$ were equal.

On trying to analyse which striker was the quickest among the strikers of $y=0.25x^2-8$, $y=7\sin(0.25\pi x)$ and $y=7\sin(0.125\pi x)$, Diana & Gisele brought the definition of speed to make sense in DG Parallel (see link K). By analysing the strikers near to zero, Gisele showed to Diana that the striker of $y=7\sin(0.25\pi x)$ was quicker than the others. Diana was still associating 'A is in front of B' with 'A is quicker than B'. At this point, Diana really revised the association. Gisele brought the idea of speed constructed in DG Parallel to DG Cartesian.

At the beginning, she tried to identify speed from DG Parallel in the motion of (x,y) while analysing the striker of $y=6$ (see links N and O). Later, on analysing the striker of $y=-0.25x^2$, Gisele argued that y was quicker than x. Diana added that y runs more spaces than x does (see link P).

According to diagram 4.4 the perceptions that Diana & Gisele developed in FP and in DG Parallel stayed completely separated, and in DG Cartesian formed separated groups of perceptions until the final interview.

By the presence of shape in DG Cartesian, pictorial views were used by Diana & Gisele to characterise the derivative of the strikers of y=x and y=2x. Diana argued that in the striker of y=2x the angle with the y-axis was smaller than the one with the x-axis. Unfortunately, she did not link angle to speed. On the other hand, by having difficulty in seeing angle without lines, Gisele did not consider this characteristic in Diana's description of y=x while guessing the striker. This difficulty motivated them to seek a functional correspondence for angle to distinguish the strikers of y=x and y=2x. They argued that the angle in the striker of y=2x is bigger because y of y=2x was bigger than y of y=x (see link M). This idea was generalised to the striker of y=-x. Note that the angle was associated with absolute value, which was valid only for the strikers given by 'linear' functions. They did not talk about the angle of the striker given by y=x-6, which could lead to a critical moment for this association.

As a result of the above-mentioned perceptions from different microworlds, I observed a gap in the link between the two perceptions in DG Cartesian which was filled in the final interview. One perception concerned variation, while the other concerned absolute value.

In the final interview, they linked 'the comparison of the speeds of y and x' in DG Parallel to the inclination of the graph (see link Q*). At this point, they did not distinguish inclination from curvature. In addition, they observed that two strikers with the same speed (y=x and y=x-6) should correspond to two graphs with same inclination (see link R*).

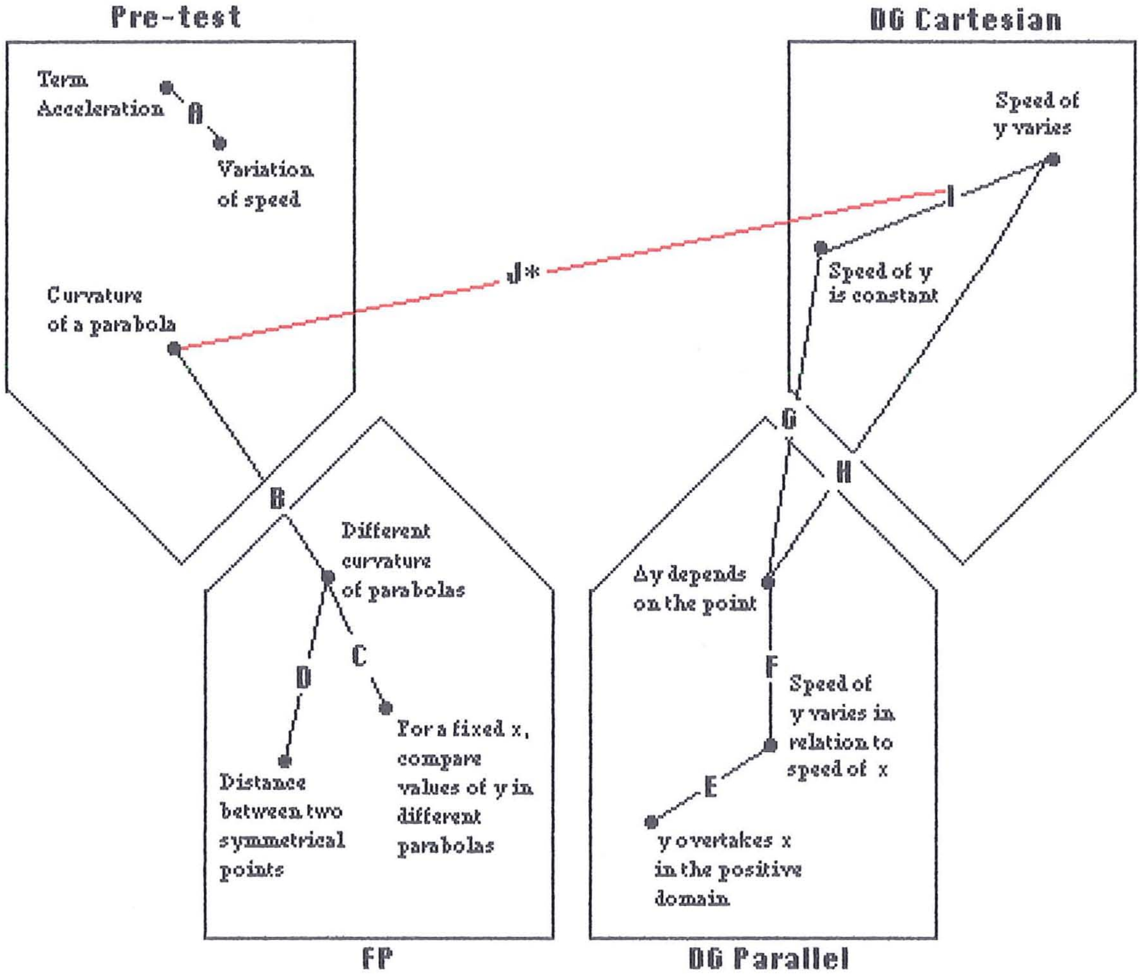
4.5 Second derivative

In the pre-test, Diana & Gisele defined acceleration as 'variation of speed' (see link A). They were not able to identify it by the equation. They also discriminated curvature of parabolas while comparing graphs without linking it to acceleration. It is interesting that they were able to compare curvature of parabolas only for parabolas with the same turning point. For instance, they argued that two translated as well as two reflected parabolas had different curvatures.

In FP Diana & Gisele discriminated curvature of graphs, while they were comparing different parabolas (see link B). On stretching the graph of $y=0.25x^2-8$ vertically, Diana & Gisele argued that this graph and the one of $y=(0.119)(0.25x^2-8)$ had different curvature. They said 'the concavity is closer than before'. Note that they always used the term 'concavity' for curvature.

Diagram 4.5

Diana & Gisele's perceptions of second derivative



On trying to distinguish parabolas by their curvature, Diana & Gisele searched for a functional perception which depended on the case. In the case of parabolas with roots at the same point, they compared the image of x for the different parabolas (see link C). For example, they argued 'for a fixed x, y [of $y=0.25x^2-8$] is around three times bigger [than y of $y=(0.119)(0.25x^2-8)$]'. In the case of parabolas with different roots, Diana & Gisele measured the distance between two symmetrical points' (see link D). Despite being different ways of measuring curvature, both perceptions are attempts to see an idea that is linked to variation in a pointwise way.

By exploring the idea of curvature while stretching vertically the graph of y=7sin(0.125 x) to the one of y=(-0.667)(7sin(0.125 x)), Diana generated a critical moment for the above-mentioned perception of curvature. She noticed that the curvature could change keeping one distance 'between two symmetrical points' fixed. Then, they revised the association between curvature and 'distance between two symmetrical points'.

While analysing curvature, Diana & Gisele also tried to compare 'values of y and x' to see which one was the biggest. This idea paralleled their perception of slope in linear graphs. Unfortunately, in doing it to two parabolas obtained by a vertical stretch, they did not realise that this characteristic changes even for the same parabola. This should have been a starting point for linking it to non-null curvature. By arguing that there is no angle in parabolas, Diana stopped a process that could have reached a link between curvature and variation of slope.

The perception of second derivative, like that of derivative, developed by Diana & Gisele in FP and DG Parallel stayed completely isolated. In the case of FP, the students' perceptions of second derivative were pictorial. In addition, on trying to relate it to a functional one, Diana & Gisele treated the idea in a pointwise way. In contrast, the perception constructed in DG Parallel by these students was based on the variation of the variables. Moreover, Diana & Gisele followed a continuous development of the idea from DG Parallel to DG Cartesian. Therefore, they reached the separation between constant and variable derivative.

In DG Parallel the idea of second derivative was observed by Diana & Gisele for the strikers which 'y overtakes x'. They observed this idea in the positive domain of the striker given by $y=0.25x^2$, arguing: "when it is coming [from zero to positive], it [y] goes slowly, slowly, so it arrives here [around x=3] it overtakes x and is quicker [than x]". Although Diana had discriminated variation of speed in the striker of $y=0.25x^2$, Gisele discriminated the speed of $y=0.25x^2-8$ as "y is always quicker than x". This perception was not deeply explored in DG Parallel.

The idea of variable speed was mentioned by Diana & Gisele again only while classifying the strikers of $y=0.25x^2-8$, $y=7\sin(0.125\pi x)$; $y=7\sin(0.25\pi x)$. At this time, the variation of y was calculated by the students to decide which striker was the fastest. When characterising the group composed by $y=7\sin(0.25\pi x)$, $y=7\sin(0.125\pi x)$ and $y=0.25x^2-8$, Diana & Gisele realised that "the strikers change speed and the fastest one would depend on the point" (see links E and F).

In DG Cartesian the students introduced the idea of variation of speed that was constructed in DG Parallel (see links G, H and I). By exploring this idea in the strikers of $y=0.25x^2$ and $y=2x$, Diana & Gisele realised that the first striker changes speed while the other one moves always one step. Then, discussing the speed of both strikers, they first compared the variation of y and x for each striker (see link I). So, they concluded that the first striker is slower near x=0 and it is quicker after x=0.

It is interesting that the pictorial perception of curvature was not observed in DG Cartesian. It seems to be hard to distinguish curvatures without trace of the graph. From a mathematical viewpoint the absence of shape allowed the students to analyse a functional idea of second derivative variationally in different functions. This analysis promoted in the students an idea of separating strikers with constant speed from strikers with variable speed. It seems to be a very constructive process of building an idea which did not suffer from barriers imposed by their previous knowledge.

Diana & Gisele's perception of curvature was very close to the one of turning point. Nonetheless, when asked in the final interview if they could distinguish a straight line from a curve without turning point they affirmed that they could. Moreover, the students linked it to constant or variable variation of the strikers when they compared strikers to define which corresponds to the parabola, without seeing the turning point. Gisele concluded that the striker of a straight line runs in regular steps while the striker corresponding to a graph with curvature runs in irregular steps (see link J*). A feature from DynaGraph used only in the final interview helped this synthesis: the dots that the strikers left.

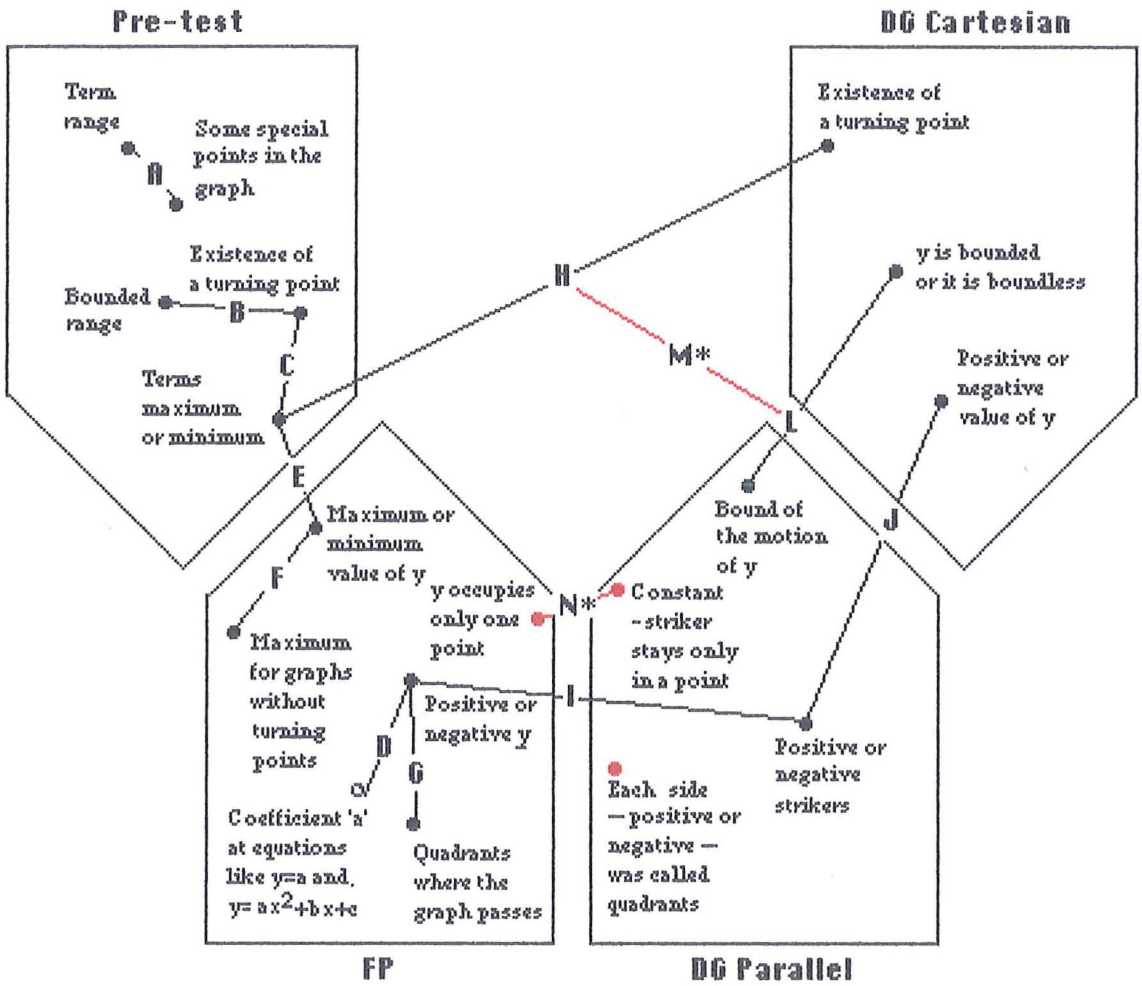
4.6 Range

In the pre-test, Diana & Gisele considered only discrete points as range. Diana argued that range is 'the point of graph where x and y meet each other' and on locating range in graphs she gave special points, such as turning point, y-intercept and x-intercept. She also gave the point (3,1) as range of y=3/x. While interpreting range in graphs, Gisele gave a collection of points (see link A). Moreover, she was not sure if range relates to y or x.

According to diagram 4.6, Diana & Gisele developed their perceptions of range in three parallel approaches in the research environment: the first approach considered 'bound of range' recognised by 'presence of turning point' in graph; the second approach considered a tendency to polarise perceptions by dividing range into positive and negative; the third approach was a perception articulated in isolation within both DG microworlds, which involves 'bound of the motion of y'. The first and third approaches are corresponding ones in different representations, which were not linked until the final interview.

Diagram 4.6

Diana & Gisele's perceptions of range



The perceptions developed by Diana & Gisele in the research environment were not linked to the term 'range'. This term used by their school stayed isolated in their pre-test. Moreover, the perceptions from the pre-test had a pointwise approach while the other perceptions related to the whole domain and range of a function.

In FP range was discriminated by Diana & Gisele by dividing it into positive and negative functions. They determined the quadrant where a graph is, then whether the function was positive or negative (see link G).

This polarisation was the basis of the association between 'positive range' and 'coefficient 'a' is positive' presented by Diana & Gisele while examining graphs of FP (see link D). While comparing the graphs of $y=6$ and $y=0.5x^2$, Diana argued that "they are similar because they were all positive". Later, she explained that because of the coefficient 'a', in both equations 'a' was positive.

From the pre-test the terms 'maximum' and 'minimum' were used by Diana & Gisele associated to turning points (see link C). They did not link them to 'minimum value that y can reach'. On exploring the transformations of graphs in FP and comparing different graphs they concluded that maximum or minimum does not depend on x, they depend only on y (see link E). For instance, exploring the graph of $y=7\sin(0.25\pi x)$, Diana & Gisele realised that all turning points of maximum have the same height.

FP was important when they generalised bounded range to graphs without turning point such as graphs of constant functions. By investigating the above-mentioned perception and by wishing to transform the graph of $y=0.5x^2$ into the one of $y=6$, Diana generalised the idea of maximum and minimum to this last graph — a graph without turning point (see link F).

In the starting activity with DG Parallel, Diana & Gisele discriminated the idea of range by 'the bound of the motion of y'. They characterised the strikers of sines by 'place where the strikers can move'. This idea was also used by Diana to 'reject' strikers that do not correspond to Gisele's description of $y=7\sin(0.125\pi x)$. Therefore, Diana generalised this perception of range to the striker of $y=x$. She considered 'absence of bound in the motion of y' arguing that 'the striker moves in all the axis'.

The tendency to divide the ideas into positive and negative also appeared in Diana & Gisele's perceptions of range in DG Parallel (see link I). They argued that the striker of $y=0.25x^2$ was only positive. At this point, Diana explained that there are three kinds of strikers: those which stay only in one side, those which stay in the middle and those which move in all the axis. By this characterisation, the range of the strikers of constant functions were considered to be different. Diana argued that the striker of $y=-3$ was similar to the striker of $y=6$, while Gisele said that "they were not because one was positive and the other was negative". Another barrier caused by this approach was in generalising this perception of range to the striker of $y=0.25x^2-8$ which was considered to be similar to those that move all over the axis. Later, they joined it to the strikers of sines but changing the characterisation of range to "in the negative side, the strikers go only up to a point — -8 or -7".

Unfortunately, in DG Parallel the approach to range that considers 'bound of the motion of y' lost importance in Diana & Gisele's work after they discriminated the striker of $y=0.5x^2$ and $y=0.25x^2$ as being only positive. This other approach was motivated by their attempt to make sense of quadrants in DG Parallel. On the other hand, the first approach was used to characterise the strikers with limit out of zero.

Gisele generalised bounded range as 'the striker can move up to some point' from the striker of $y=7\sin(0.125\pi x)$ to the one of $y=0.25x^2-8$.

The idea of range was not emphasised in DG Cartesian by Diana & Gisele. They looked at range mainly using the positive and negative approach (see link J). They characterised the striker of $y=-0.25x^2$ as "y does not pass to the positive side". Gisele felt very clearly that it was y that was not going to the positive side. This perception of range was also attributed by Diana to the range of the striker of $y=0.25x^2$.

On the other hand, to distinguish the striker of y=-x from the striker of parabolas, Gisele used the idea of 'bound of the motion of y' (see link L) which the students brought from DG Parallel. After realising that they had mismatched the strikers of y=-x and $y=-0.25x^2$, Gisele generalised the idea from the strikers of parabolas to 'y is not bounded' to the striker of y=-x.

Another perception which re-appeared in DG Cartesian was extreme values. They started using the term minimum again associated with turning point. Diana & Gisele considered minimum to describe the strikers of $y=0.25x^2-8$ and $y=0.5x^2$.

Despite using the term quadrants in FP to determine 'place through which a graph passes', Diana & Gisele saw these quadrants in a very special way. Diana stated that "I thought it [quadrants] was a mathematical rule, I didn't think that when a point was at the first quadrant was when x and y is positive, this one [fourth quadrant] that when x is positive and y is negative...". These actions demonstrated an interpretation of graph in a pictorial not in a functional way.

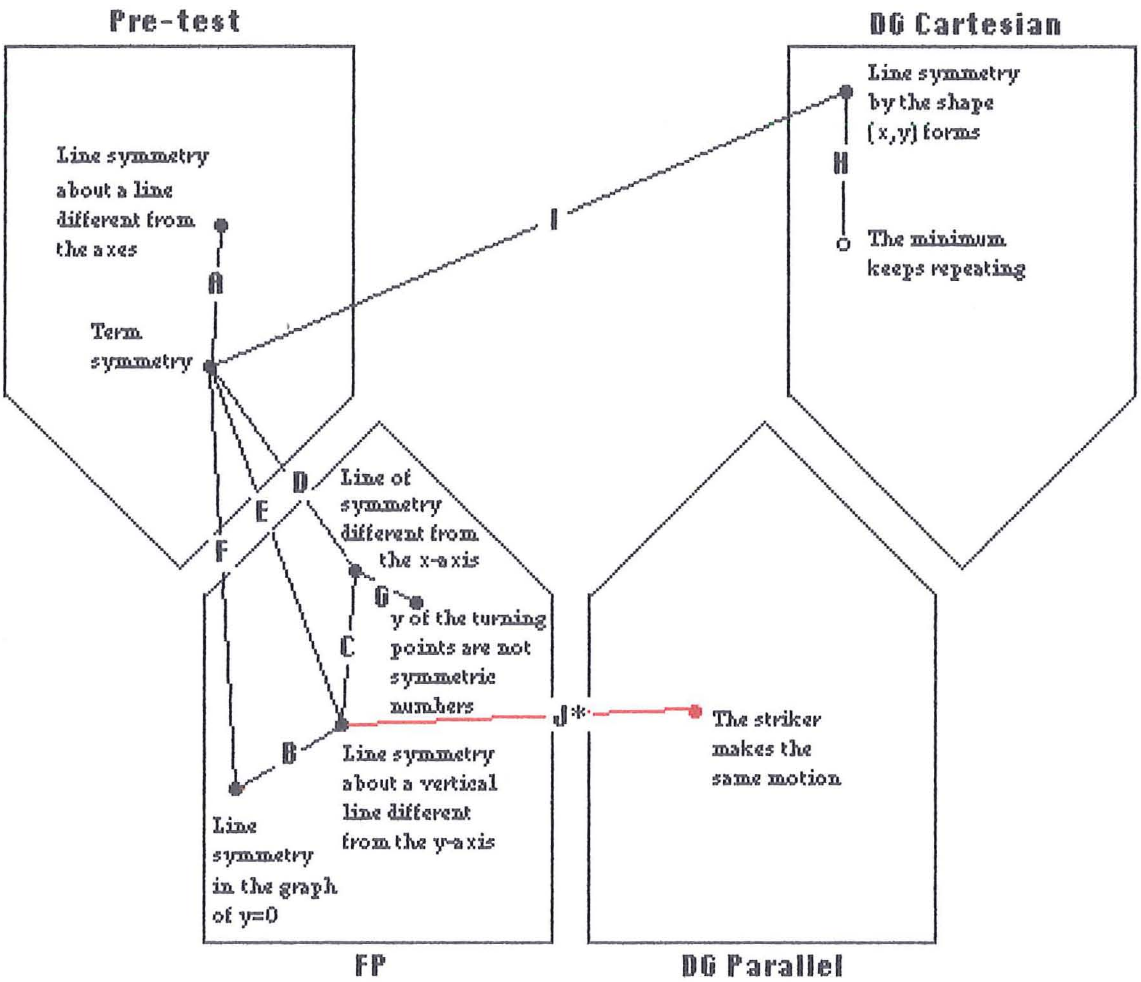
In the final interview the link between the two other approaches to range was not straightforward. Diana & Gisele reached the link between 'bound of the motion of y' of DG Parallel and 'space of the y-axis that a graph occupies by existence of turning point' (see link M*). Firstly, on being asked to distinguish the strikers corresponding to the graphs of $y=-0.25x^2$ and $y=-0.25x^2+10$, they matched the turning point to the 'point where x meets y'. Secondly, they argued that "y would go up to 10 and return while x will be at 0". Later, on comparing two sines vertically stretched, they recognised that the new striker would move a bigger interval based on the turning points. On the other hand, using a vertical translation in a graph of sine, they could not predict what would happen to the new striker. On looking at the strikers in DynaGraph, they realised that 'the places where the strikers move' were different but they had the same amplitude. To conclude, these students were able to link 'bound of the motion of y' to their perception of extreme values by turning point. Nonetheless, Diana & Gisele were not able to localise ranges in graphs of translated sines.

Moreover, the students linked the idea that the constant graph occupies only one point to the idea that the striker stays only in a point (see link N*) — two new perceptions.

4.7 Symmetry

Diagram 4.7

Diana & Gisele's perceptions of symmetry



As shown in diagram 4.7, since the pre-test Diana & Gisele were familiar with line symmetry. They did not limit their perception to symmetric numbers or to line symmetry in the axes. They identified line symmetry in the graphs by tracing the lines of symmetry (see link A). However, this perception was expressed only pictorially in the graph. Therefore, as diagram 4.7 demonstrates, Diana & Gisele identified line symmetry in all microworlds that present the Cartesian representation.

In FP they identified firstly that the graph of $y=0.25x^2-8$ was line symmetric to the graph of $y=-0.25x^2+11.6$ after a vertical reflection. Moreover, they argued that their turning points have different absolute value because the line of symmetry was at y=1.8 (see link D). Note that at this stage they noticed that the association between line symmetry and symmetric numbers was not useful for these parabolas (see link G). Secondly, after reflecting the graph of $y=-0.25x^2$ into the one of $y=-0.25(x+14)^2$, Diana localised the line of symmetry in both graphs (see link E). Thirdly, Gisele localised the symmetry of the graph given by $y=7\sin(0.25\pi x)$ as being any vertical line passing through a turning point. Despite being able to determine all sorts of line symmetry, in FP Diana & Gisele did not try to investigate a pointwise or a variational corresponding idea.

The interaction with FP was important to the students' exploration of a canonical symmetry, for instance, the symmetry of a constant function. Links B and F show that on obtaining the graph of y=0 while exploring line symmetry on the graph of y=abs(x) using the vertical stretch, these students generalised line symmetry to the constant function. The line of symmetry was placed on the y-axis by Diana. This point was the turning point of y=abs(x). They did not observe that this line of symmetry could pass through any point. This evidences some association between line symmetry and turning point.

As a result of the absence of shape in DG Parallel, Diana & Gisele did not explore any symmetry in the strikers. In FP, they discriminated only line symmetries, not symmetric numbers.

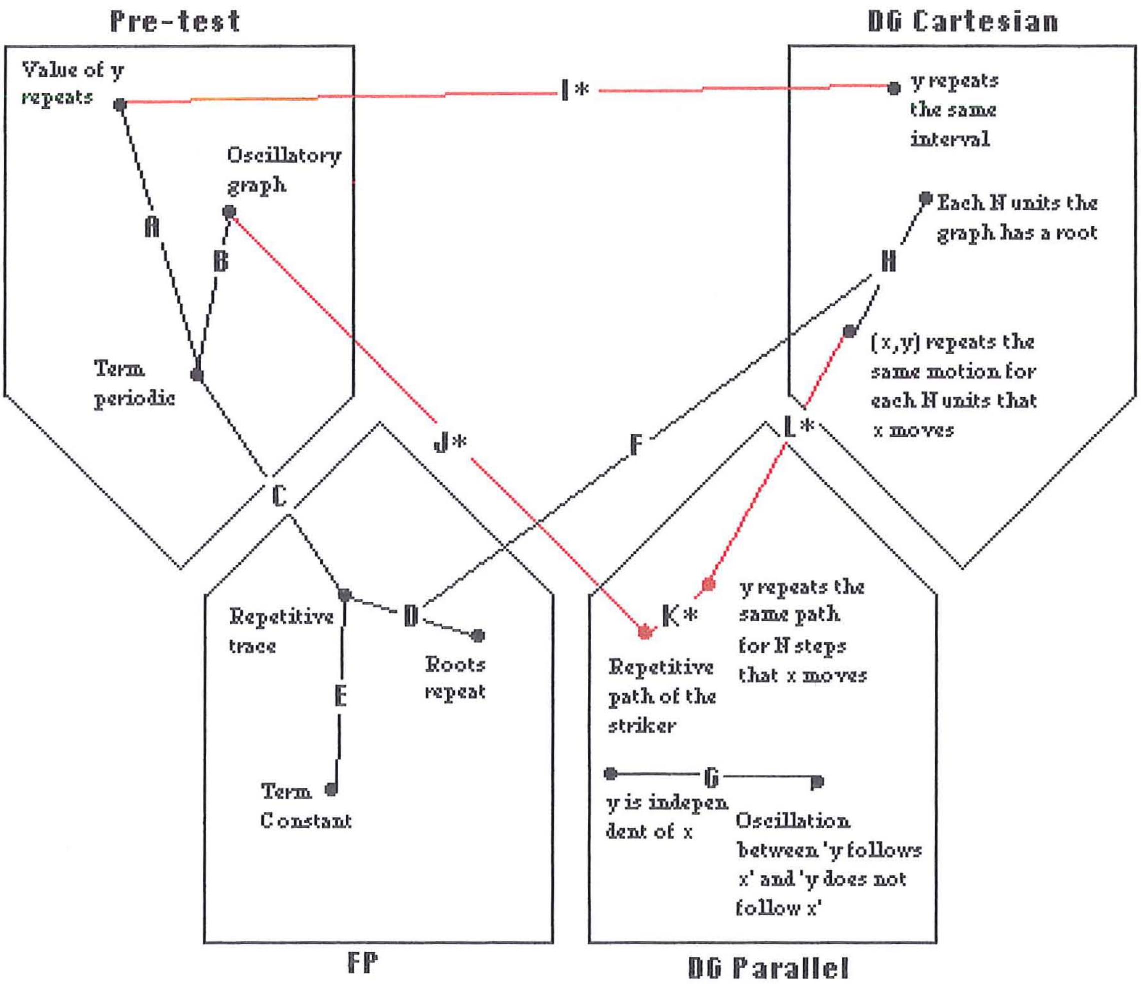
The work with DG Cartesian encouraged the students to seek a pointwise equivalent to the idea of line symmetry. Unlike in DG Parallel, the idea of symmetry was explored in DG Cartesian by Gisele while working with the striker of $y=7\sin(0.25\pi x)$. This exploration is due to the presence of shape drawn by (x,y) (see link I). Gisele recognised the striker as being symmetric by a vertical line passing through one of its turning points. Then, the absence of lines in the graph encouraged Gisele to try a pointwise correspondence for line symmetry. Unfortunately, it was related more to periodicity than to symmetry. In addition, she explored this idea only once which did not lead to progress. She explained the line symmetry in the striker of $y=7\sin(0.25\pi x)$ by: “Because it has the same points... minimum and after it repeats” (see link H).

Motivated by direct questioning in the final interview, Diana & Gisele connected the vertical line symmetry in graphs to 'the strikers make the same motion' in DG Parallel (see link J*) — a variational view. Nonetheless, on trying to explain the effects of this symmetry on a horizontal translation of the graph of $y=0.25x^2$, they were not able to identify line symmetry in the new striker.

4.8 Periodicity

Diagram 4.8

Diana & Gisele's perceptions of periodicity



In the pre-test, Diana & Gisele perceived the idea of periodic function as 'function with oscillatory path' as well as by 'the repetitive behaviour of y'. For instance, they defined periodic function as being 'function where its value always repeats' (see link A). Moreover, in the Cartesian representation Diana pointed to parabola and oscillatory graphs as being periodic. In the first graph, she mismatched periodicity with line symmetry. By a similar perception, Gisele pointed to an oscillatory and aperiodic graph as being periodic.

In FP, Diana & Gisele improved their perception of periodicity arguing that there was a constant repetition of roots and trace. They identified periodicity in the graph of $y=7\sin(0.25\pi x)$ by 'the repetition of the trace' (see link C). In addition, Diana & Gisele pointed to 'repetition of roots' to explain this characteristic (see link D). It is interesting that Gisele called this behaviour constant. As Diana understood constant as the same value of y, Gisele explained that "the trace repeats, it is always the same, it never changes" (see link E).

Considering that their perception of periodicity was based on graphs up to FP, in DG Parallel the closest idea to periodicity observed by Diana & Gisele was the oscillatory behaviour between 'y follows x' and 'y does not follow x' in the striker of $y=7\sin(0.125\pi x)$. They attributed to this oscillatory behaviour the idea that 'y is independent of x' (see link G), because, unsuccessfully, they tried to separate the domain where 'y follows x' into positive and negative.

On the one hand, Diana & Gisele's perceptions of periodicity in DG Parallel were linked to the oscillation between increasing and decreasing. Diagram 4.8 shows that this perception stayed completely isolated from the ideas they had in the other microworlds, which contain the Cartesian representation. On the other hand, the same diagram shows that the sequence pre-test, FP and DG Cartesian helped the students to reach a perception of periodic behaviour in a variational way. In DG Cartesian they also separated the behaviour of x, y and (x,y).

In DG Cartesian, Diana & Gisele perceived periodic aspects of the striker of $y=7\sin(0.25\pi x)$. They observed 'the repetition of the interval that y moves'. As in FP, in DG Cartesian they also observed in this striker 'the repetition of the path that (x,y) does' (see link H) arguing that "in four units that x moves, (x,y) was doing one 'parabola' and returning to zero" (see link F). Despite arguing that repetition was not only in roots, they always used roots to count period. They also generalised this perception to the striker of $y=7\sin(0.125\pi x)$ (see table AIV-8.2) considering that '(x,y) was doing each 'parabola' in 8 units that x was moving'.

After constructing the variational perception of periodicity in DG Cartesian, in the final interview Diana & Gisele were able to bring it to DG Parallel. They connected 'oscillation of graph' to 'repetitive path of the motion of the striker' (see links J* and K*). Moreover, the idea that '(x,y) repeats the same motion for each N that x moves' was identified in DG Parallel as 'y repeats the same path for N units that x moves' (see link L*). After observing a horizontal stretch in the graph of $y=7\sin(0.25\pi x)$, they guessed what would happen to the striker corresponding to the stretched graph.

Diana also linked the 'repetitive interval that the striker of $y=7\sin(0.25\pi x)$ moves' to the 'repetitive height the graph reached in y' which was the perception presented in their pre-test (see link I*).

5 Anne & Jane's perceptions of the function properties

Anne & Jane were the other pairs of students who followed the activities from DG to FP microworlds.

5.1 Turning point

In the pre-test Anne & Jane's perceptions of turning point were very close to the idea of extreme values. For instance, Jane defined turning point as being 'point where they can find maximum or minimum' (see link A). Although they perceived a clear separation between turning point and extreme values, a good question is: are they able to find maximum without turning point?

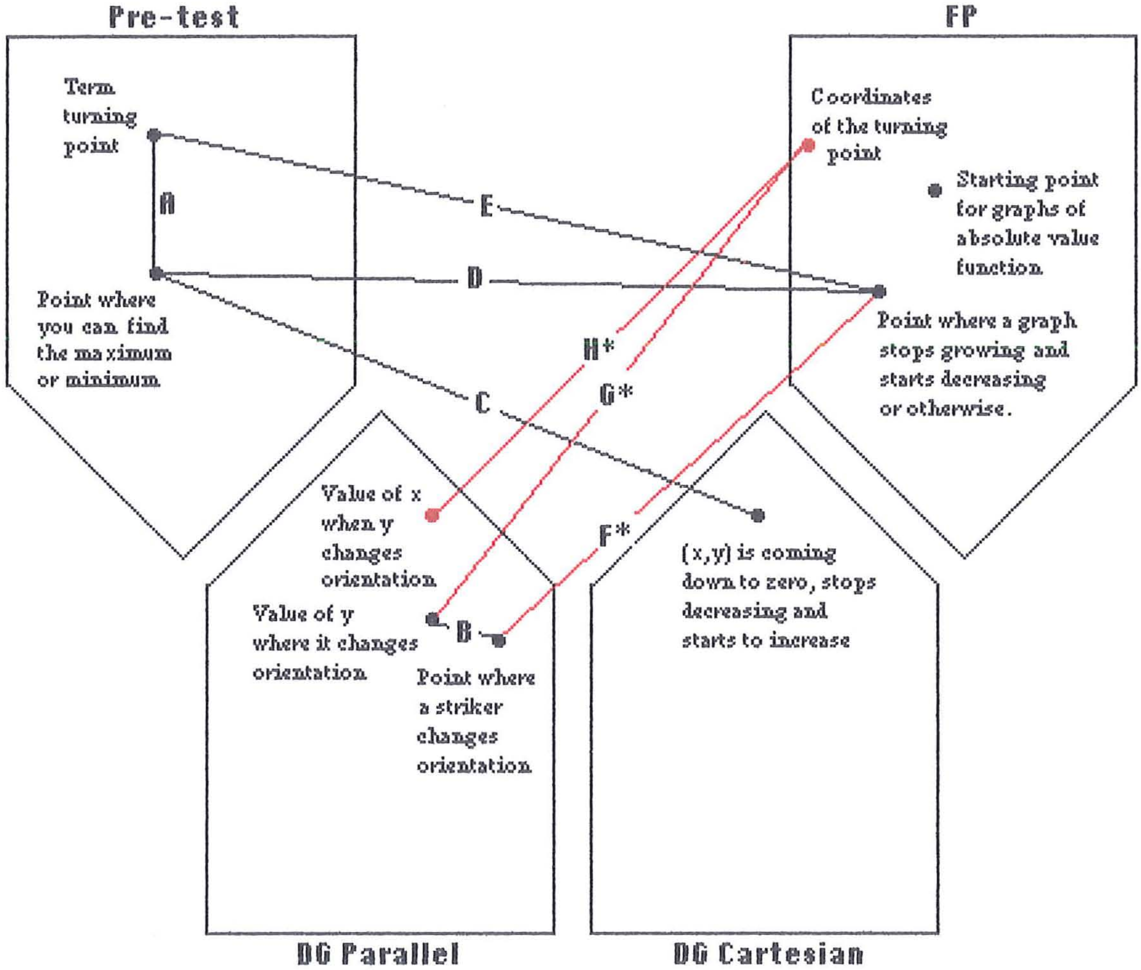
In DG Parallel turning point was discussed by Jane & Anne as being 'the point where the striker changes orientation in relation to the orientation of x'. They started taking note of 'value of y' at this point for the striker of $y=7\sin(0.25\pi x)$, then, generalised to almost all the strikers of functions with and without turning points (see table AIV-1.1). In other words, they described the strikers corresponding to linear functions as the strikers which did not change orientation.

In DG Cartesian Jane & Anne used 'the motion of the sprite of (x,y)' to recognise a turning point. Link C represents their connection between 'motion of (x,y)' and the idea of extreme values through the shape that (x,y) traces: "It [(x,y)] is coming down to zero [(0,0)], it stops decreasing and it starts to increase". This perception of turning point was used by Jane & Anne to recognise and classify the strikers as being parabolas. Note that they did not observe 'the behaviour of x and y' at the turning points.

In FP Jane & Anne presented two different perceptions of turning point: 'starting point' was mentioned by the students only for graphs of absolute value functions; 'point where a graph stops growing and starts decreasing or otherwise' was mentioned while they were discussing extreme values for parabolas. Jane explained that a graph had maximum because "the turning point was the point where it stops growing and starts decreasing or otherwise for minimum" (see links A, D and E). In section 5.6, I discussed how Jane & Anne associated turning point and extreme values.

Diagram 5.1

Anne & Jane's perceptions of turning point



It is interesting that in all the microworlds, excepting in DG Parallel, Jane & Anne's perceptions of turning point were related to extreme values. Nonetheless, they presented perceptions of turning point similar to the one they expressed in DG Parallel which stayed isolated. On the other hand, DG Parallel was the only microworld in which Jane & Anne separated variables when talking about turning point. As for the other microworlds, they treated turning point as a special point on graphs without referring to the behaviour of y or x.

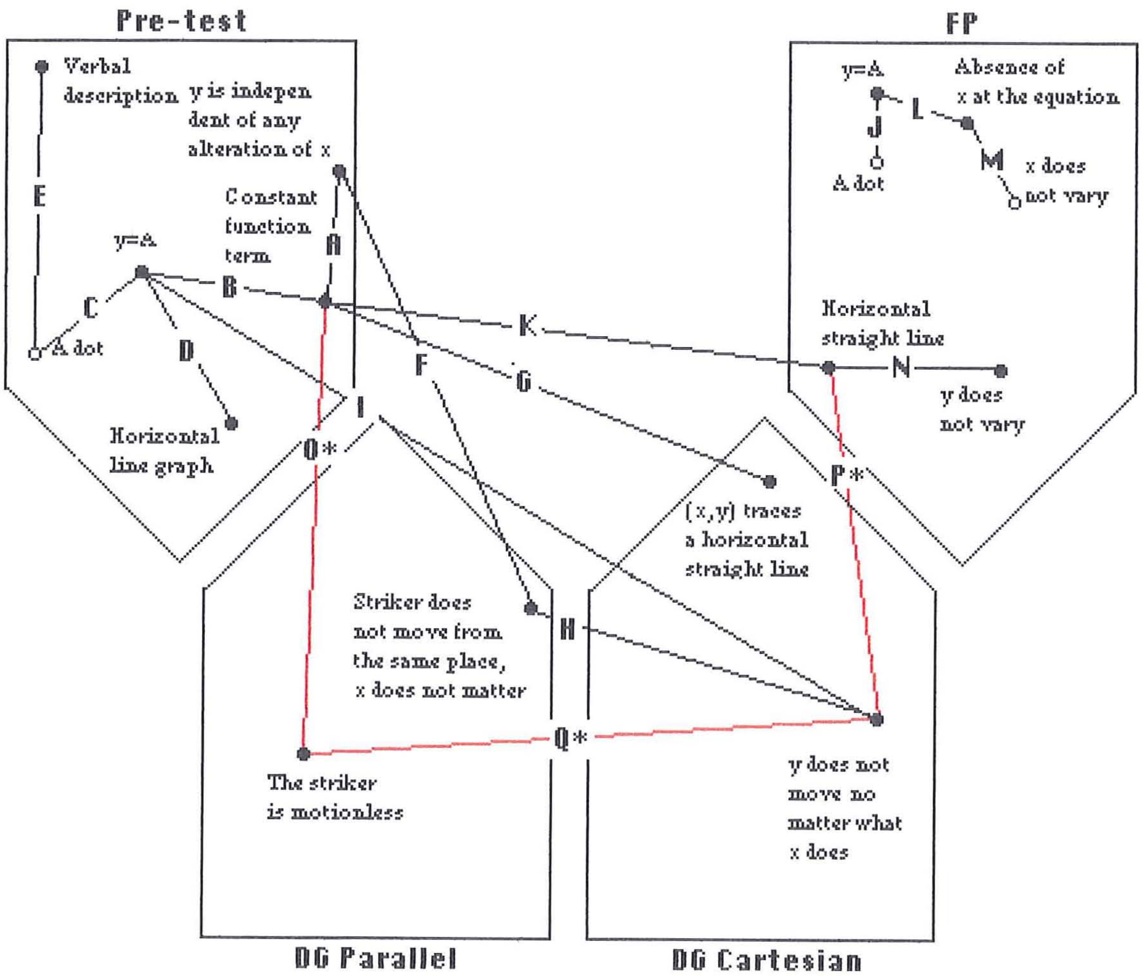
In the final interview Jane & Anne used turning point as 'the point where the graph stops growing and starts decreasing' to recognise turning point in DG Parallel as being 'point where the striker returns' (see link F*).

In the final interview the students started separating x and y in the Cartesian representation. It was a consequence of their connection between coordinates in the graph and the values of y and x in DG Parallel (see links G* and H*). They were asked to predict the behaviour of the new strikers corresponding to graphs obtained by vertical and horizontal translations in graphs of parabolas.

5.2 Constant function

Diagram 5.2

Anne & Jane's perceptions of constant function



In the pre-test Jane & Anne's perceptions of constant function varied according to the representation and the activity that they were involved in. Both students represented a motionless car as a dot in a graph of distance per time (see link E). Anne also traced the graph of y=2 as a dot (see link C). Nonetheless, Jane's definition of constant function seems to be mathematically correct (see link D). Her definitions are expressed by links A and B: y=a and 'y is independent of any alteration of x'. Meanwhile Anne perceived constant function in the algebraic representation as 'independent coefficient was equal to zero'.

Diagram 5.2 shows that in DG Parallel Jane & Anne characterised strikers of constant functions in two similar ways: 'striker does not move from the same place' and 'striker is motionless'. Despite being very similar both characterisations allowed different paths in Jane & Anne's perceptions of constant function. The first perception allowed Jane to identify the striker of a constant function by 'y is independent of x' (see link F). She argued: "There are two constant functions; x can vary how much it wants but y will be always in the same place". The second perception seems to have a special status in DG Parallel. For example, it was only for these strikers that Anne & Jane broke their criterion 'y was over x at zero' to group the strikers creating a separated group for the motionless ones. The second perception stayed isolated in DG Parallel up to the final interview.

In DG Cartesian Jane & Anne discriminated the strikers of constant functions in two ways, as in their perception from DG Parallel. The first one considered 'motions of x and y', while the second perception dealt with 'motion of (x,y)'. As regards the first perception, Anne & Jane constructed links H and F with previous knowledge and DG Parallel using 'y is independent of x' while describing the striker of y=6. Using this perception, they also constructed the equation for this striker (see link I). Talking of the second perception, they started to describe the constant function by the shape (x,y) traces. Link G shows that Jane & Anne matched this shape with the term constant function. Despite discriminating both perceptions, Jane & Anne did not link 'behaviour of x and y' and 'motion of (x,y)' in DG Cartesian. The only relation between these sprites observed by Jane & Anne was 'point where y meets (x,y)' — a special point. Therefore, DG Cartesian was not used as a bridge between DG Parallel and Cartesian representation of constant function by Jane & Anne. However, the variational perception was linked in the final interview.

Despite being articulated in terms of the microworlds, 'y is motionless' was linked to the term 'constant function' in the final interview by matching strikers with graphs (see link O*). A similar connection was made by Jane & Anne when asked about the corresponding idea of horizontal straight line in DG Parallel. Nonetheless, this link passed through the idea of independence (see links P* and Q*). First, Jane & Anne linked 'horizontal straight lines' to 'y does not vary, only x varies'. Then, they connected the last-mentioned perception to 'y is motionless, while x can vary'.

Regarding the interpretation of constant function through its equation, it seems that 'absence of x' at the equation represents a difficulty in their interpretation. For instance, when dealing with equations in FP, Jane & Anne returned to the same perception of its graph as a dot. Moreover, there is a gap between their perception of 'what does not vary' — x or y — through the graphic and algebraic representations. Despite thinking of 'variation of x' while looking at the equation (see link M), the students pointed to y as being 'the variable that does not vary' while looking at the graph (see link N). It is interesting that, as in the pre-test, in FP Jane & Anne were not able to recognise a constant function by its equation. Nonetheless, they quickly linked the term 'constant function' to the shape of its graph. For instance, as soon as they took the equations on board, Jane & Anne imagined its graph as being a dot by 'the absence of x' (see links L and J). By tracing it in FP, they argued that it was a constant function (see link K). It is interesting that 'the perception in which y does not vary' started to be discussed after they stretched the graph of $y=2x$ into the one of $y=0$. This indicates an influence of the interaction with dynamic transformations of graphs in Jane & Anne's perceptions of constant function.

5.3 Monotonicity

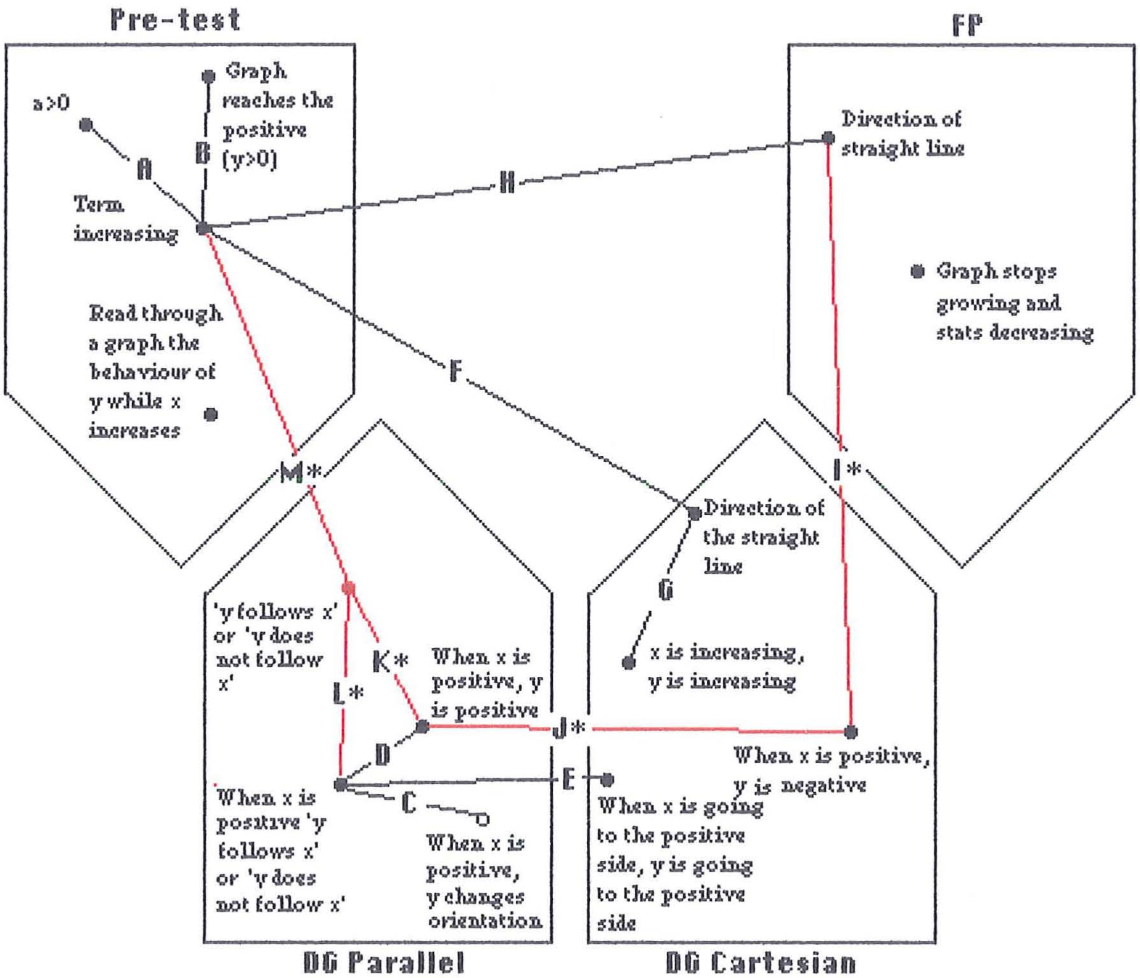
It is interesting how Jane & Anne associated the terms 'increasing' and 'decreasing' with some polarised rules arising from the pre-test which are, in general, valid only for linear functions. For instance, Anne's definition of the term 'increasing' was linked to linear coefficient: "when $a>0$" (see link A). Jane's definition emphasised polarisation when analysing graphs: "increasing is a function that reaches positive value at the system ($y>0$)" (see link B). In spite of all these associations, the students were able to interpret a graph variationally when they were asked about the behaviour of y when x increases or decreases. Therefore, this is evidence of an obstacle linked with the use of mathematical terminology at school.

Diagram 5.3 shows two kinds of perceptions the students had of monotonicity. The first is connected with the term 'increasing' which reflects their previous knowledge about monotonicity. The second group of perceptions are variational. In DG Parallel, this second perception enabled Jane & Anne to generalise the idea for other kinds of functions such as parabolas.

In DG Parallel, the students discriminated monotonicity by looking at the positive and negative parts of the domain separately. For example, as they chose to analyse x in the positive side first, the striker of $y=-x$ was considered to be different from the striker of $y=0.5x^2$.

Diagram 5.3

Anne & Jane's perceptions of monotonicity



By analysing the striker of y=x-6, Jane replaced 'when x is positive, y follows x' for the polarised rule 'when x is positive, y is positive' (see link D). By discussing with Anne the same example, Jane revised her own association. She showed that despite being on different sides, the striker moves to the same side. The same argument was used by Anne to generalise the perception to $y=0.25x^2-8$. The analysis of the strikers given by sines also offered to the students a critical moment to overcome this polarisation. Nonetheless, they only classified the striker of y=7sin(0.25 x) as "the striker changes many times" (see link C).

The polarised approach was more important to them than the analysis of other functional characteristics. Jane & Anne adopted two criteria for classifying the strikers: the striker is zero when x is zero and in the positive domain 'y follows x' or 'y does not follow x'. It was only when analysing the group of the strikers of y=x,

$y=2x$, $y=0.25x^2$ and $y=0.5x^2$ that they separated the strikers which change orientation from those which do not.

As in DG Parallel, in DG Cartesian Jane & Anne presented two distinct perceptions. The first perception being 'y follows x' or 'y does not follow x' with the polarisation of domain, which was articulated within DG Parallel, was brought to DG Cartesian. On analysing the behaviour of x and y only, they changed 'y follows x' into "when x is going to positive, y is going to the positive" (see link E). The second perception is linked with their previous perception of monotonicity by direction of the straight line traced by (x,y) (see link F). Note that this perception reduced the sample in which the students generalised monotonicity to linear functions. This suggests that this link created a barrier for generalising the idea among other kinds of functions. Note that despite being similar perceptions in the same microworld, they were not linked.

Although the perception of the term 'increasing' was confined to linear functions, the students gave a variational interpretation for the property in DG Cartesian. The absence of a trace of a graph encouraged them to seek a functional correspondence to 'direction of straight line' (see link G). This perception was also presented in the pre-test, but was not linked to the term 'increasing', staying completely isolated.

It is interesting that at the end, while subdividing the group composed by the strikers of $y=x$, $y=2x$, $y=-x$, Jane & Anne used the variational correspondence of 'direction of straight line' to subdivide the group. While explaining why the strikers of $y=x$ and $y=2x$ were together, Jane said: "both [strikers] are increasing, x is increasing, y is increasing".

In FP the students used 'direction of straight lines' to recognise whether a function was increasing or decreasing. For instance, after stretching horizontally the graph of $y=x$ into the one of $y=-x$, Jane argued that these functions had different directions, which were connected by Anne to the terms 'increasing'and 'decreasing' (see link H).

It is interesting that while exploring extreme values, Jane & Anne interpreted the graph of $y=-0.25x^2$ as increasing or decreasing. Nonetheless, they did not link it to the terms 'increasing' and 'decreasing' from school knowledge. Moreover, they did not separate the behaviour of x and y while analysing growth.

In the final interview Jane & Anne connected 'direction of a straight line' to 'y follows x' or 'y does not follow x' (see links I*, J* and L*). They also connected this perception to the term 'increasing' or 'decreasing' restricted to straight lines (see link M*). These two connections passed through an association that appeared in DG

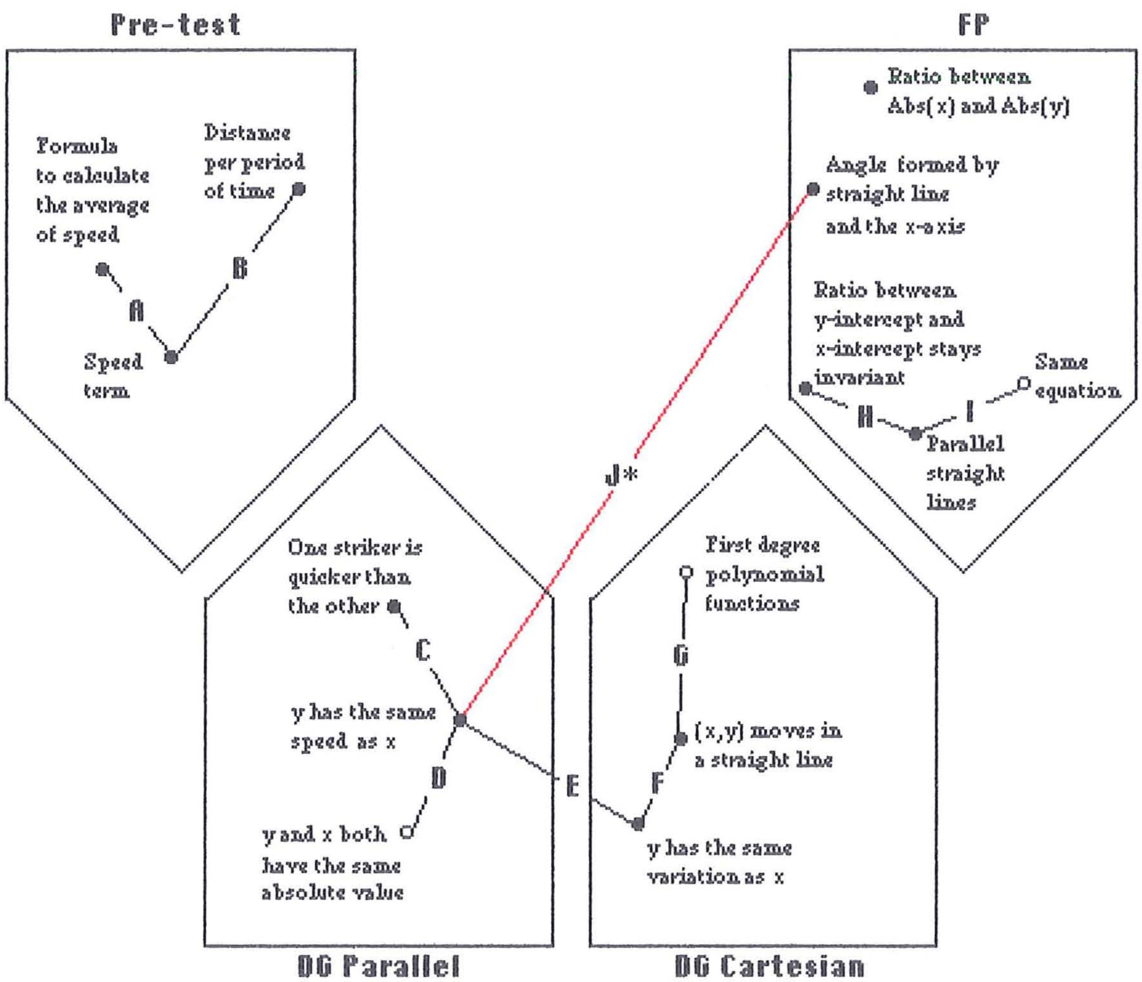
Parallel as well as in DG Cartesian. In order to achieve the above-mentioned syntheses, the students identified 'strikers that move in only one orientation' as being 'straight lines'.

On trying to generalise the connection to the striker of y=-0.25x^2, they used the rule 'when x is positive, y is positive' to mean 'y follows x' (see link K*). Once more the polarised rules were strong in these students' perceptions of monotonicity.

5.4 Derivative

Diagram 5.4

Anne & Jane's perceptions of derivative



In the pre-test Jane & Anne did not use slope to interpret derivative in a graph. For instance, in the question about the cyclists, Anne tried to build an equation from the graph to calculate the speed. Unfortunately, she only knew the formula to calculate average of speed (see link A). Both students' perceptions of speed while defining are related to distance per time (see link B).

Derivative was discriminated by Jane & Anne from the starting activity with DG Parallel. During a long period of their work Anne & Jane characterised the striker as quick or slow, writing: "the striker [of $y=0.25x^2-8$] is quick".

Only when comparing the strikers of y=2x and y=x, Jane realised that she could compare their speeds with the one of x. Firstly, their perception was 'y has the same speed as x', which was generalised for the striker of y=-x (see link C). Secondly, by analysing the striker of y=x-6, Anne associated this perception with 'y and x have same absolute value' (see link D). Nonetheless, by discussing Jane's argument "the distance that y moves is the variation", Anne revised this association. This phase was the beginning of their findings of constant and variable derivative.

Anne & Jane constructed in a continuous way their perception of derivative as being 'comparison of the variations of x and y' from DG Parallel to DG Cartesian (see link E). They were not observing the motion of (x,y). On noticing it, Jane & Anne achieved their major findings on DG Cartesian. While classifying the strikers, they asked if the strikers of first degree polynomial function had a fixed variation (see links F and G). This finding will be discussed in the next section.

The horizontal stretch in the graph of y=abs(x) encouraged Jane & Anne to establish a way to measure slope of the graphs. First, Jane constructed the idea of derivative by the internal angle. On stretching the graph of y=abs(x) horizontally, Anne explored the idea by measuring the abs(x)/abs(y). By stretching vertically the graph of y=x into the graph of y=-x, both students sparked off their curiosity about 'inclination of straight lines'.

The perception of derivative continued to be explored when they tried to explore the idea of parallelism. By translating the graph of y=x vertically, Jane noticed that this command was keeping invariant 'the inclination' as well as 'the ratio between y-intercept and x-intercept' (see link H). At first, Anne had considered two parallel straight lines should have the same equation (see link I). These ideas were originated in the students' curiosity as to whether the inclinations of two straight lines were the same. Thus, the interaction with FP was responsible for the students matching 'parallelism between two straight lines' with 'the same ratio between y-intercept and x-intercept'.

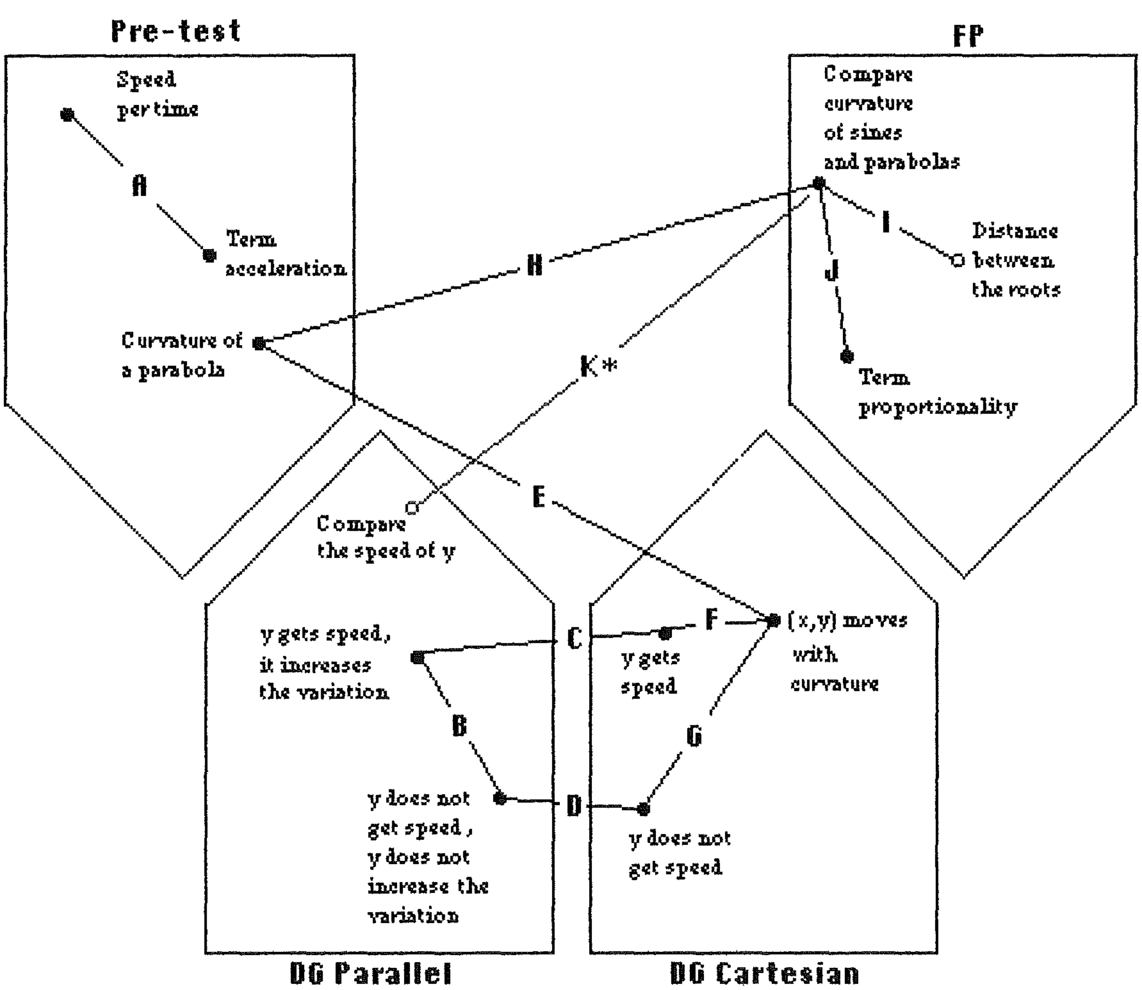
In addition, Jane & Anne perceived derivative by the angle formed by a straight line and the x-axis. Despite being similar to the perception presented in DG Cartesian, there was no evidence of spontaneous connections between these perceptions.

Anne & Jane linked 'inclination of straight line' to 'speed of strikers' while comparing different linear functions in both microworlds (see link J*). Nonetheless, this link was not straightforward. Firstly, while analysing the strikers of y=x and y=-x, they argued that inclination was given by 'distances between x and zero and between y and zero', that is, these distances should be the same. They were analysing the strikers of y=x and y=-x. The analysis of the striker of y=x-6 made them switch to another rule — the distance between x and y stays the same. When analysing the striker of y=-x again, they observed that the rule was not valid. Suddenly, they stopped and Anne argued "it is the speed! the speeds of y are the same".

5.5 Second derivative

Diagram 5.5

Anne & Jane's perceptions of second derivative



As to the idea of derivative, in the pre-test the students did not use graph (curvature) to interpret second derivative — acceleration. For instance, Anne transformed graphs into equations to calculate acceleration, while Jane argued that the acceleration of a parabola is zero because its speed is zero. They had defined acceleration as speed per time (see link A).

On the other hand, the students discriminated curvature in a graph. They presented difficulties in measuring curvature in graphs. By comparing curvature, for example, Anne pointed out that two vertically translated parabolas had different curvature. Meanwhile, she pointed out that two parabolas differing by a vertical reflection and a horizontal translation had the same curvature.

Diagram 5.5 shows that Jane & Anne's perceptions of second derivative were all linked from their previous knowledge to FP. These links were in general made using the graphic representation of second derivative.

In DG Parallel, Anne & Jane constructed the idea of variable derivative which was generalised later to constant derivative. On comparing the measure of the variations of the strikers of $y=0.5x^2$ and $y=2x$, Anne concluded that $y=0.5x^2$ "gets speed, it varies the variation" (see link B). While classifying the strikers, Jane & Anne subdivided the group composed by $y=0.5x^2$; $y=0.25x^2$; $y=2x$; $y=x$ into those strikers which 'y increases the variation' and those strikers 'y did not get speed'. Unfortunately, they were not able to generalise this idea to strikers of of sines. This suggests that the students used "to get speed" meaning that 'y leaves the screen speeding up' like the striker of quadratic functions.

Jane & Anne brought from DG Parallel to DG Cartesian the idea of variable and constant derivative. For instance, while comparing the strikers of $y=0.25x^2$ and $y=0.5x^2$, they argued that the striker of $y=0.5x^2$ varies the variation a lot (see links C and D). Nonetheless, as in other properties, the students did not pay attention to the behaviour of (x,y).

Only after noticing that (x,y) made a turning point (see links F and G), they started to classify the striker in the families of functions from their school knowledge. At this point, a critical moment happened for them to link 'y gets speed' and '(x,y) moves with curvature'. Anne argued that the striker of $y=0.25x^2$ could be an absolute value. This doubt led the students to link E which was not straightforward. First, they compared the last-mentioned striker to the striker of $y=2x$. Then, they noticed that (x,y) of the first striker moves in a curve. Second, to decide whether (x,y) of the striker of $y=0.5x^2$ has a bending or straight movement, they remembered that they had distinguished the strikers as: 'y gets speed' and 'y does not get speed'. Finally, Jane & Anne observed from the strikers that 'when y gets speed, (x,y) moves bending'. Moreover, while observing the striker of $y=7\sin(0.25\pi x)$, Jane generalised variable speed to this striker. That is, she argued that 'y gets speed' because there was curvature in the motion of (x,y). Jane doubted this, saying that it could be oscillatory by straight lines.

As diagram 5.5 shows, the distinction between 'straight line or curve' and 'constant or variable speed' was not directly linked to their perception in FP. The links were made through the pre-test (see link H). In FP the idea of curvature was used in a pictorial distinction of the graphs of $y=0.25x^2$ and $y=0.5x^2$. The students used to say that 'one graph was more closed or more open' than the other graph. This perception was generalised by them to the graphs of $y=7\sin(0.25\pi x)$ and $y=7\sin(0.125\pi x)$. This passage marked a special moment that revealed a way they used to measure curvature: Jane explained: "by the distance between the roots".

A continuous transformation between the graph of $y=0.25x^2-8$ and $y=0.25x^2$ promoted a critical moment for Jane & Anne to revise the above-mentioned perception of curvature. While translating the graph of $y=0.25x^2-8$ towards the one of $y=0.25x^2$, Jane argued that "the curvature was becoming smaller". Then, she added: the command would change the curvature of the parabolas. In doubt, she noticed that the command was not modifying the curvature. It was modifying only 'the distance between the roots' (see link I).

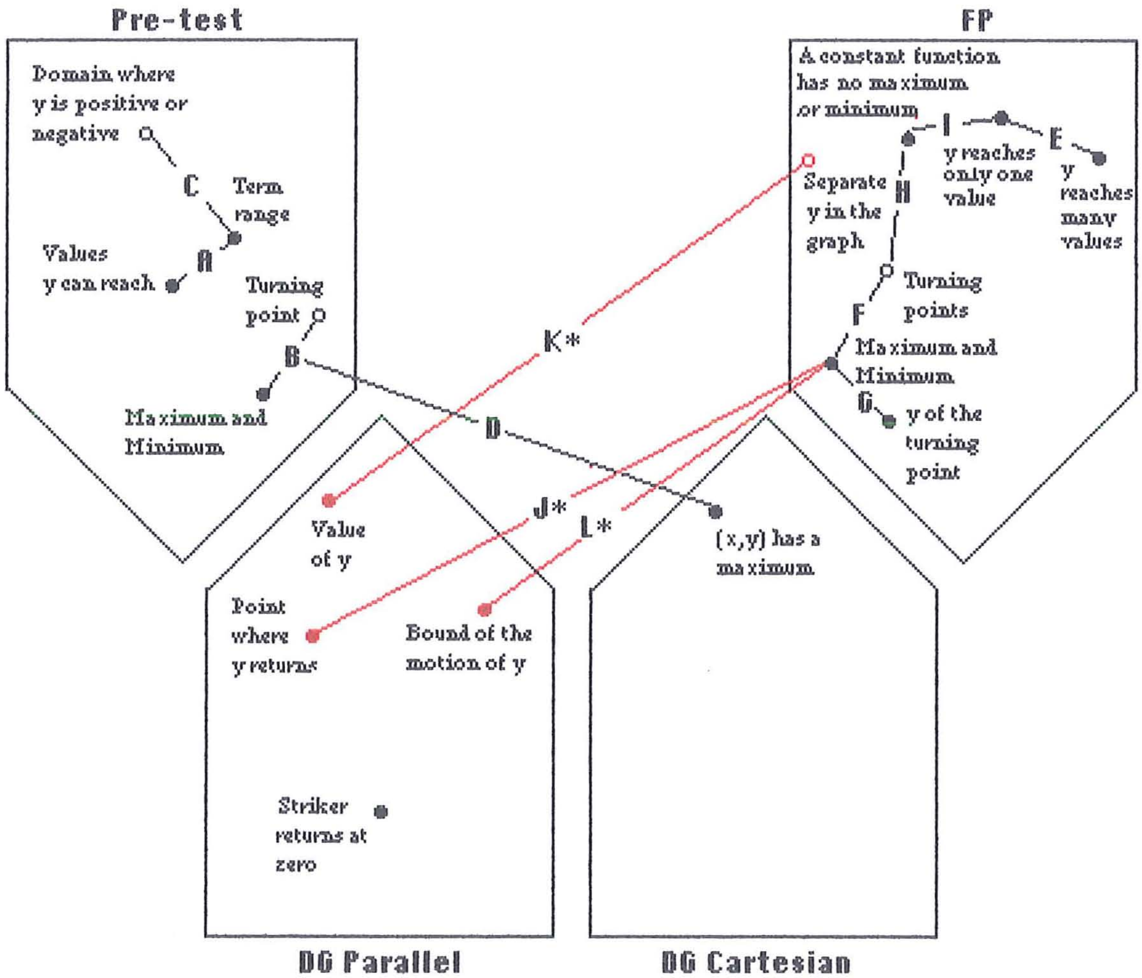
This interaction with dynamic transformations of graphs in FP created in the students the need for measuring the curvature of a graph. It is interesting that in DG Parallel they did not feel the necessity to measure the variation of the rate of change. By discussing the curvature of the graph given by $y=0.25x^2$ using the vertical translation, Jane & Anne constructed a perception of curvature of a parabola which they called 'proportionality of a parabola' (see link J). They argued that while moving up or down a parabola you never change the 'proportionality' of a parabola. It seems more interesting that the use of the last mentioned command scaffolds a method of comparing the curvature of two parabolas. They started to put one turning point on the other in order to compare the trace of the graph. In other words, they promoted a vertical translation from one parabola into the other one.

As in Jane & Anne's development through the research environment, in the final interview they linked 'change of variation of y' to 'curvature of graph'. Nonetheless, while comparing the curvatures of the strikers given by $y=0.25x^2$ and $y=0.5x^2$, they compared the 'speeds of y' (see link K*), instead of 'accelerations of y'. This difference in variable speed of strikers seems very hard to measure as is curvature of graph.

5.6 Range

Diagram 5.6

Anne & Jane's perceptions of range



In the pre-test Anne perceived range differently from Jane. Anne considered it as being 'value which y can reach' (see link A). In contrast, Jane assumed a polarised approach to the term range. Her definition of range divided the domain for which y is positive and for which y is negative — what she herself called "the study of the sign" (see link C) referring to the topic from which she took her approach. Despite having a definition of range incorrect from a mathematical viewpoint, Jane followed her definition while identifying range in graphs. In contrast, Anne knew the definition of range but she was not able to discriminate it in a graph.

In the pre-test, Anne interpreted extreme values only for a graph with turning point (see link B). For instance, she could not find minimum or maximum of a graph of a constant function and a hyperbole.

In DG Parallel, Jane & Anne discriminated hardly any property related to range. Only in the starting activity with DG Parallel Jane discriminated "the point where the striker [of $y=-0.25x^2$] returns".

As in DG Parallel, in DG Cartesian Jane & Anne did not work much with the idea of range. While analysing the striker of $y=-0.25x^2$, they distinguished it from the one of $y=0.25x^2$ by: 'the first has maximum, the other has minimum'. They also used extreme values to classify the turning points of the strikers given by sines (see link D).

In FP Jane distinguished the range of the graphs given by $y=-3$ and $y=-x$ as 'y reaches only one value' and 'y reaches many values' (see link E), respectively. This is also a reason why the students argued later that $y=6$ had no maximum or minimum (see link I).

When using the graphs of $y=7\sin(0.25\pi x)$ and $y=7\sin(0.125\pi x)$, Jane argued that by turning points 'these functions had same maximum and same minimum' (see link H and F). From this statement together with their perception that a constant function had no maximum or minimum, I observed that their perception of maximum or minimum was associated with the existence of a turning point. Moreover, they treated maximum and minimum and turning point as having the same meaning.

An evidence of their above-mentioned association as well as a critical moment in revising it was the use of horizontal translation in the graph of $y=0.25x^2$. Jane argued that the graph was changing its minimum. Meanwhile Anne, who had already linked maximum to 'y of the turning point' by using the vertical translation (see link G), interposed saying that "the maximum was y of the turning point, it was changing the turning point" not the minimum. Therefore they separated the idea of turning point from the idea of maximum while exploring the dynamic commands of FP. On the other hand, it was not separated completely from the existence of a turning point.

This perception of extreme values made the students distinguish the graphs with maximum and minimum from the graphs with maximum or minimum. In other words, the parabolas were separated from the sine graphs by the limits of their ranges.

The polarised perception of range appeared only when Jane & Anne were working with a vertical translation in the graph of $y=7\sin(0.125\pi x)$. They translated it to get a graph with positive range.

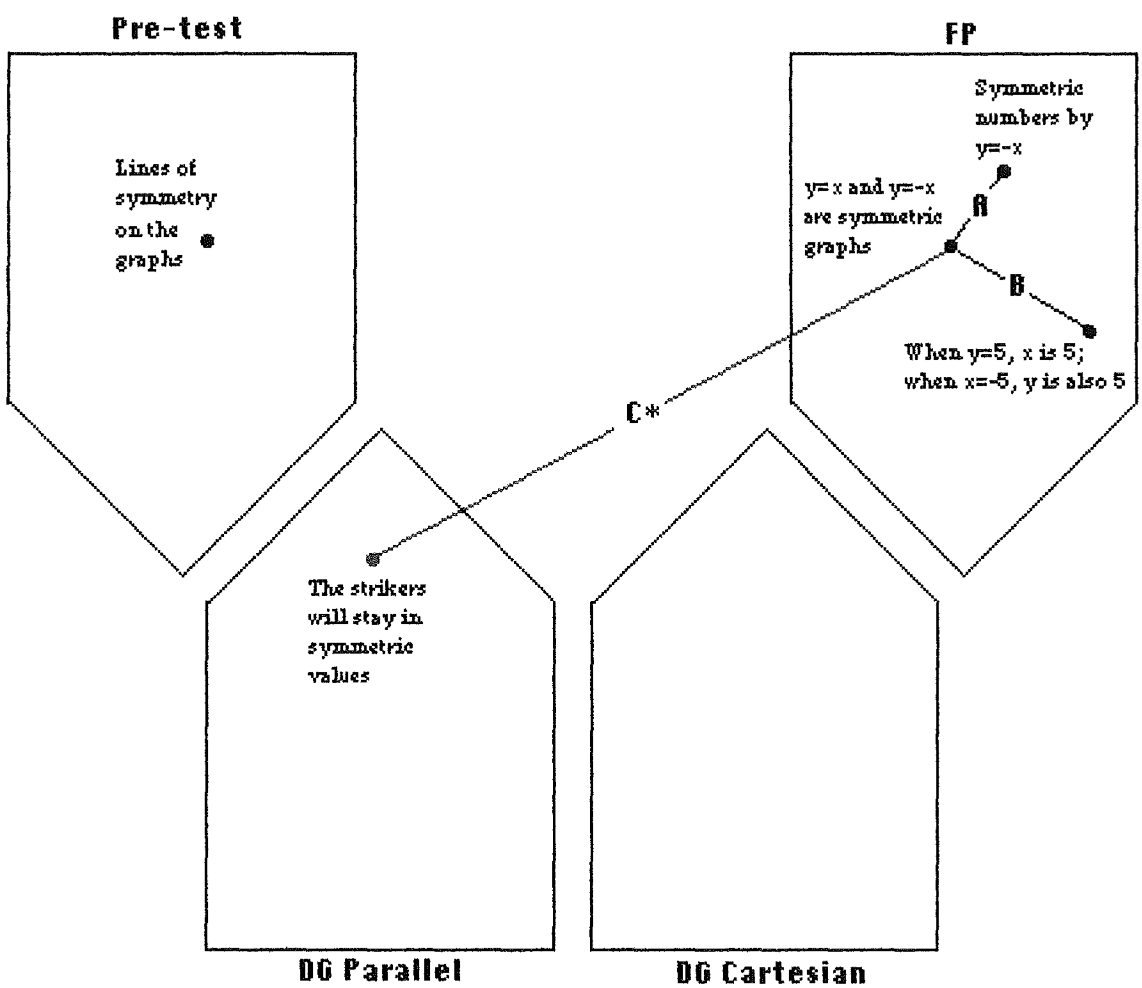
In the final interview these students linked 'value of y in a graph' to 'place where a striker was'. In other words, they identified the output of each point in each representation (see link K*). Moreover, 'bound of place where a striker can move' was connected to 'bounded range of graph'. They did this in order to decide whether the striker of y=7sin(0.25πx) corresponds to a graph of sine (link L*).

Regarding the idea of extreme values, Jane & Anne connected it as being 'point where striker returns' (see link J*). That is the same synthesis as their perception of turning point. Moreover, they were not able to distinguish in this synthesis when it is maximum or when it is minimum. They tried rules like 'if the striker stays in the positive side the point is minimum'; polarised rules which are valid for very few samples of functions.

5.7 Symmetry

Diagram 5.7

Anne & Jane's perceptions of symmetry



Despite recognising in the pre-test any line symmetry in graphs, Jane & Anne hardly explored this idea throughout the research environment. Even in FP, they only discussed line symmetry between the graphs of y=x and y=-x. Nonetheless, on trying a pointwise correspondence for their idea, they limited it to line symmetry in one of the axes, which has correspondence to symmetric numbers. Symmetric numbers were also discriminated by Jane & Anne in FP. Link A shows that Jane discriminated symmetric numbers at the equation of y=-x in a first contact with the equations. Later, in order to explain the symmetry between y=x and y=-x, Anne tried a similar perception in the graphs associated with symmetric numbers. Jane explained: "when y=5, x is also [5]". When trying to guess the function, Anne completed "in the other graph when x=-5, y was also [5]" (see link B).

In the final interview, on being asked about the symmetry between y=-x and y=x in their strikers, Jane & Anne corresponded it to 'strikers having the same speed'. After seeing a counter-example of their link obtained by a vertical translation in the graph of y=x, the students reviewed their link. Moreover, they linked it to "one striker will be at one value while the other will be in the symmetric value" (see link C*). Note that unlike link B, link C* corresponds to line symmetry in the x-axis. Nonetheless, it does not seem that the students perceived the difference.

5.8 Periodicity

In the pre-test, only Jane tried to define periodicity. She considered to be periodic "those functions that don't have considerable modifications in their path" (see link A). She also added a sentence to exclude the constant functions. In addition, with this definition Jane mismatched periodic graphs with symmetric graphs. She considered a parabola as being periodic (see link B).

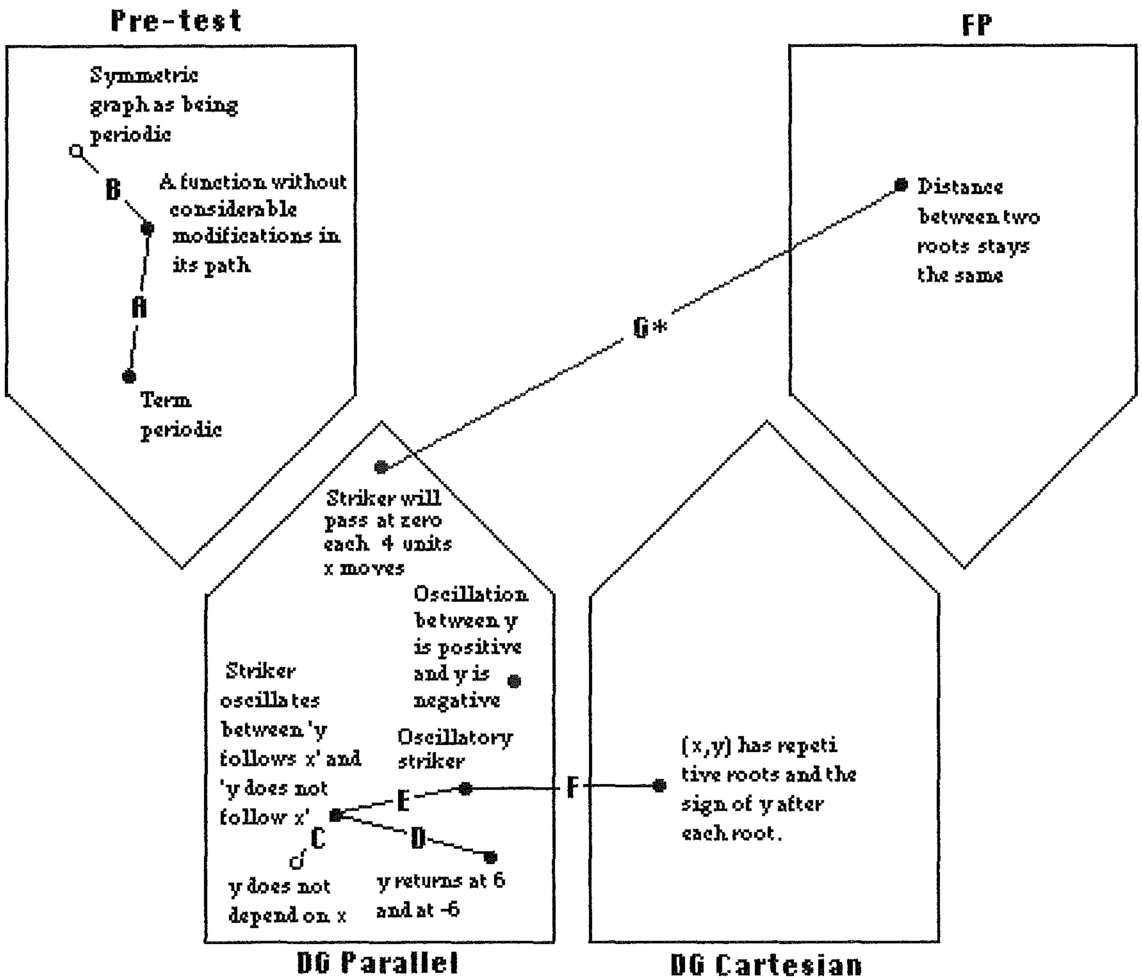
In DG Parallel Anne & Jane presented a barrier to the construction of the idea of periodicity. It was due to the oscillation between 'y follows x' and 'y does not follow x' within the polarised approach with which they analysed the idea of monotonicity. They confused it at first with the idea that 'y is independent of x' (see link C).

Afterwards Jane & Anne overcame the barrier perceiving the motion of the striker of $y=7\sin(0.125\pi x)$ as oscillatory. This conclusion was motivated by observing the repetitive behaviour between 'y follows x' and 'y does not follow x' (see link E). This observation led Anne to take note of the value where y returns (see link D).

As diagram 5.8 shows Jane & Anne established a continuity in building the idea of periodicity from DG Parallel to DG Cartesian. The students started taking note of the values of the roots and the signs of y after each root. Then, Jane & Anne sketched the graph of the striker. From this sketch they observed that the strikers with repetitive roots corresponded to the oscillatory strikers (see link F). Note that they never reached the approach of periodic oscillation.

Diagram 5.8

Anne & Jane's perceptions of periodicity



In FP 'the distances between the roots from the graphs of y=7sin(0.25 x) to y=7sin(0.125 x)' was the unique aspect of periodicity explored by Jane & Anne. Note that this is an aspect based on special points.

Jane & Anne used many polarised characteristics such as positive or negative. In the final interview they also used oscillation between positive and negative to match the strikers of sines with the graphs. Then, 'periodicity of roots' was linked to 'striker passes at zero each four units that x moves' (see link G*) by the use of special points. This was the first time they went further than oscillation.

VIII — Cross-sectional Analysis

In this chapter, I will compare the findings from different pairs of students and make a comparison with the school approach to functions. First, the cross-sectional analysis of the evolution of perceptions of each function property will be presented, then, the results under the headings: synthesis, associations, obstacles, and the influence of each microworld on the students' perceptions of function.

1 The evolution of students' perceptions of each function property

This section will present the similarities and differences in the findings concerning students' perceptions of each of the properties investigated in the different microworlds.

1.1 Turning point

Initially both Diana & Gisele and Jane & Anne viewed turning points as extreme values, which coincides with the way this idea is introduced in the school mathematics. The other two pairs perceived turning points as special points in the Cartesian system linked to parabolas. They all used turning point and curvature to recognise a parabola. Only Charles among all the students perceived a turning point as 'point where the graph changes direction'.

In DG Parallel by analysing the motion of y while moving x, all the pairs of students perceived a turning point in a variational way — as being 'the point where the striker changes orientation'. Both pairs of students who began by working with FP used DG Cartesian as a bridge between this variational perception and their idea of turning point in the Cartesian representation. For the other pairs of students this perception remained isolated in DG Parallel until the final interview when it was linked to the turning points in graphs. This suggests that the previous work in FP encouraged these students to try to match strikers with graphs. In doing so, they linked up the idea of turning point after recognising the shape of the graph in DG Cartesian.

The pairs of students who began by working with DG concentrated their observations on the motion of (x,y) in DG Cartesian without considering the motions of x and y. The shape formed by (x,y) suppressed their observations of x and y from DG

Parallel. Nonetheless, when these students were asked to correspond the ideas, they were able to connect the variational perceptions of turning point derived from activities in DG Parallel to their previous perceptions of turning points in graphs.

Diana & Gisele and Jane & Anne did not use dynamic transformations of graphs to revise their perceptions of turning point. It seems that their previous knowledge of turning point suppressed subsequent perceptions (see table AIV-1.3). In contrast, the other pairs used the transformations in different ways to revise their previous knowledge as well as to develop new perceptions of turning point. This revision varied according to the idea they were investigating as well as the examples they were working with. For instance, turning point as 'point where the graph changes from increasing to decreasing' was perceived in graphs with curvature only when Bernard & Charles were stretching the graph of $y=-0.25x^2$ horizontally, or when John & Tanya were exploring the graph of an absolute value. This suggests that the exploration of these transformations affords critical moments for the students to revise their previous knowledge and to create different perceptions of the property. These critical moments were explored in some cases by the students but not all — the counter-example of the association between maximum and turning point generated by Diana & Gisele did not provoke them to revise their previous 'school' perception.

Two pairs of students identified 'the point where y meets x' as special points in DG Parallel. These points were connected with turning point in graphs when they tried to correspond the properties from graphs to DG Parallel. Turning point as well as 'points where graphs cross the axes' are special points observed by these students while analysing the graphs of parabolas.

1.2 Constant function

The motionless behaviour of y is a property with 'special status' in DG Parallel characterising constant functions. All the pairs of students used the motionless behaviour to describe and classify the strikers of constant functions. The emphasis on the motion (or variation) of x seemed to be the main reason for this kind of perception. The fact that all the students classified the constant functions as a group isolated from the linear functions in DG Parallel showed that, unlike in the Cartesian representation, in DG Parallel the strikers of constant functions were considered as completely different from those of linear functions.

Another important aspect developed by the students in DG Parallel deals with the notion that 'y is independent of x'. The possibility of 'varying x and observing the behaviour of y' led them to observe constant function by seeing that 'y is independent of x'. By noticing that they could not alter y by moving x while exploring the strikers given by y=-3 and y=6, they perceived an independence of x.

It is interesting that this perception completely articulated within the microworld was connected to other perceptions of constant functions throughout the research environment by all the pairs. Only Anne & Jane kept it isolated until the final interview. Nonetheless, links P*, Q* and O* of diagram CVII-5.2 show that these students were able to connect this perception throughout the microworlds.

Three of the pairs of students used DG Cartesian as a bridge between a variational perception of constant function from DG Parallel and its Cartesian representation. In DG Cartesian, all the pairs explored 'the behaviour of (x,y)' and 'the relation between the behaviour of x and y'. All the pairs of students, apart from Jane & Anne, connected those two perceptions in a spontaneous way. Jane & Anne worked in DG Cartesian building perceptions connected to their previous 'school' knowledge. They also connected these perceptions to their previous knowledge of constant function as a horizontal straight line.

At least one student from all the pairs, except Bernard & Charles, sketched the graph of y=2 as a dot — (0,2) in the pre-test. Bernard & Charles did not explore the algebraic representation of constant function. The pairs of students who began by working with FP gradually came to perceive 'the absence of x' in the equation as corresponding to 'y is independent of x' in the graph by exploiting the continuous feedback between graph and equation while transforming graphs in FP. Jane & Anne, one of the pairs of students who began by working with DG, returned to the same perception in FP despite building the equation of the striker of a constant function in DG Cartesian. Nonetheless, Anne & Jane connected all their perceptions of constant function which were related to graphs, but those related to equations were isolated (see diagram CVII-5.2). In the final interview, their perceptions were connected through 'y does not vary'. In conclusion, FP was used to develop the perception of constant function as 'y does not vary', and connected with the 'absence of x' in the equation by three of the pairs of students.

A different perception of constant function was expressed by the pairs of students who began by working with FP. They discriminated the constant functions as a step between increasing and decreasing functions. This suggests that the students used FP to perceive increasing, constant, and decreasing as steps of a continuous variation of derivative. Note that this is a break in the compartmentalisation of the perception of constant, monotonicity and derivative, which seems to be more difficult to perceive in DG Parallel.

1.3 Monotonicity

The students' previous knowledge of monotonicity exhibited limitations derived from the fact that the school only emphasised this property for linear functions. All the pairs of students recognised this property as a characteristic of straight lines by 'the direction of straight line'. Diana & Gisele and Jane & Anne presented the link between 'sign of linear coefficient' denoted by 'a' and the terms 'increasing' and 'decreasing'. Diana & Gisele over-generalised the meaning of 'increasing' for parabolas by linking 'sign of angular coefficient' also denoted by 'a', 'positive or negative curvature' and these terms. In contrast, all the students were able to analyse monotonicity in a graph without using the terms 'increasing' and 'decreasing'. All the pairs of students, apart from John & Tanya, analysed the graph of y=3/x in a variational way in the pre-test. John & Tanya did this only in FP. These contrasts suggest that their previous knowledge linked to the term increasing caused an obstacle when the students tried to generalise the property for non-linear functions.

In addition, the attempts that the students made to give a functional meaning to the term 'increasing' followed rules which involved polarisation and only worked for linear functions. For example, Diana & Gisele linked the term increasing to the rule 'after x-intercept, y is positive' in FP.

In contrast to these barriers, in DG Parallel all the pairs of students discriminated monotonicity by comparing the orientation of the motions of x and y. Although these perceptions remained confined to the microworld interaction until the final interview for all the pairs, they were able to generalise the perceptions to strikers given by parabolas in which the rules did not work. It is interesting that in the final interview, all the pairs linked the terms 'increasing' and 'decreasing' to 'orientation of the motions of x and y' for linear functions. Bernard & Charles and Diana & Gisele used 'orientations of the motions of x and y' to generalise the meaning of the terms 'increasing' and 'decreasing' for quadratic functions, considering that these functions change between 'increasing' and 'decreasing'. Meanwhile the attempt of the other two pairs to achieve this synthesis was blocked by their previous and persisting perception of the term 'increasing' as 'the direction of the graph'.

Although the students were able to build the above-mentioned perception of monotonicity in DG Parallel, they all tended to associate the idea with polarised rules. For example, as a result of this tendency, apart from Jane & Anne, all the students interpreted the strikers of sines as being 'y is independent of x'.

In DG Cartesian the students interpreted monotonicity as: (a) 'the orientations of the motions of x and y' disconnected from the terms 'increasing' and 'decreasing' and generalised to all the functions and (b) 'the directions of the graph traced by (x,y)' linked to these terms and restricted to linear functions. All the pairs of students, apart from Bernard & Charles, presented both interpretations. Nonetheless, only Jane & Anne linked both. Note that they were also the only pair who used (a) only for linear functions. Bernard & Charles presented only interpretation (b). Therefore, DG Cartesian worked as a bridge between DG Parallel and Cartesian system only for Jane & Anne.

When using microworlds with Cartesian representation, all the pairs perceived monotonicity as 'the direction of straight line'. Table AIV-3.3 shows that while exploring stretches, three of the pairs revised their perceptions of monotonicity. Bernard & Charles, for example, used the horizontal stretch to connect the ideas of slope and monotonicity. Another interesting point was John & Anne's generalisation of a variational perception of monotonicity that they built in FP to non-linear graphs. Note that the only students who did not use transformations of graphs to explore monotonicity were Diana & Gisele who presented a persistent over-generalisation of the term increasing as being 'a>0'[1] for any kind of function.

1.4 Derivative

All the pairs of students used the speeds of x and y to characterise the strikers of linear functions. Moreover, all pairs except John & Tanya generalised this property to the strikers of non-linear functions which indicates a positive aspect of the interaction with DG Parallel. Even John & Tanya tried to investigate it, but on matching the strikers with graphs and on linking angle and 'ratio', these students gave up. Thus, the link created a barrier to this generalisation. These results suggest that derivative as speed has a special status in DG Parallel.

Diagram AIV-4.1 shows that all the pairs of students, apart from John & Tanya, used DG Cartesian as a bridge between perceptions built in DG Parallel and those built in FP. For John & Tanya these connections were made when they tried to match the strikers and the graphs interacting with DG Parallel. All the pairs of students identified the perceptions of derivative derived from interactions with DG Parallel in DG Cartesian by (a) 'comparing the motions of x and y'. They all except John & Tanya

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[1] 'a' was always used by the students to denote: 'the linear coefficient' of linear functions given by y=ax+b, 'the angular coefficient' of quadratic functions given by $y=ax^2+bx+c$ and the coefficient of sine functions given by y=asin(bx)+c. These notations were also presented in the textbook.

also discriminated (b) 'the inclination of the graph traced by (x,y)' in DG Cartesian. Bernard & Charles used DG Cartesian as a bridge to connect the perception (b) to FP. The other pairs used these two perceptions of derivative differently. Diana & Gisele, who began by working with FP, perceived (b) isolated from (a) linking both sets of perception in the final interview. Meanwhile, Anne & Jane, who began by working with DG, spontaneously linked the perceptions in DG Cartesian. Therefore, all the pairs of students connected their perceptions of derivative as speed and as angle formed by the straight line and the x-axis.

Although all the students knew the definition of speed in the pre-test and used speed in both DG Parallel and DG Cartesian, only Diana & Gisele used the definition in DG Parallel. Moreover, for the other three pairs of students, the perception built in a continuous and connected process seemed to have replaced the perceptions presented in the pre-test.

Although in DG Parallel all the students achieved the variational perception of derivative, they started by associating speed with pointwise perceptions. Initially, they compared speeds as quicker or slower. After that, they compared the speeds of x and y. Nonetheless, these speeds were measured by a 'ratio between absolute value', instead of by variation. Finally, on generalising the perception to the striker of y=x-6, all the students revised the perception to consider 'variations of x and y'. Bernard & Charles were the only pair who did not do this until their final interview.

Looking at the students' perceptions in FP allowed me to see how many ways the derivative of linear functions can be seen: inclination of the graph (angle formed by a straight line and the x-axis); linear coefficient; ratio between values of x and y; ratio between y-intercept and x-intercept. It is interesting that the last perception was built while exploring vertical translations in linear graphs. According to table AIV-4.3 after exploring this command, three of the pairs of students observed that this ratio was invariant and linked it to the inclination of the graph as well as to its linear coefficient. For the other students, the use of this command was linked with parallelism. They over-generalised a 'ratio between absolute values of y and x' to affine graphs. On the other hand, the use of the stretch commands encouraged all the students to analyse the derivative as 'ratio between values of x and y'. Therefore, with regard to derivative there was an established pattern between perceptions generated in the exploration of stretch and of translation. It is interesting that only Jane & Anne presented isolated perceptions of derivative in FP. The other students linked all their perceptions within FP.

1.5 Second Derivative

In the pre-test the students treated second derivative as acceleration and curvature. All the pairs knew the definition of acceleration as a variation of speed. Nonetheless, none of them were able to interpret acceleration as the curvature of a graph. They traced graphs without distinguishing straight lines from curves. Therefore, there was no previous link between curvature and acceleration. None of the pairs linked angular coefficient of quadratic equations to curvatures of graphs.

The idea of variation of speed was used by all the pairs of students to characterise the functions (strikers) in DG Parallel. John & Tanya used it only in the starting activity in DG Parallel. The other pairs of students compared speeds or 'ratio between the values or the variations of y and x', concluding with the separation of the strikers which vary this ratio from those which do not. Two of these pairs reached this separation in DG Cartesian while only one was able to do it in DG Parallel.

Diagram AIV-5.1 shows that, apart from John & Tanya, all pairs of students brought the perceptions from DG Parallel to DG Cartesian and linked them to their previous perceptions of curvature. One of the big issues of this research was that the interaction in the sequence DG Parallel to DG Cartesian enabled all the pairs, apart from John & Tanya, to classify 'the ratio between the variations of y and x' as constant and variable. Apart from Diana & Gisele, the other two pairs also linked this classification to the separation between straight lines and curves. Diana & Gisele made this connection in the final interview. Therefore, they built a variational perception of the second derivative distinguishing a constant from a variable derivative. John & Tanya also constructed this classification and this link but only in the final interview. They had to overcome a barrier to this link created by their associations of a 'ratio between x and y' and the angle in the graph.

The pairs who started by working with DG used DG Cartesian as a bridge between the perceptions built in DG Parallel and their previous knowledge. Jane & Anne perceived second derivative in DG Cartesian as 'the variation of the speeds of y and x' and 'the shape traced by (x,y)' and they linked these perceptions in FP. For Charles & Bernard, the perceptions as 'the variations of the speeds of x and y' built in DG Parallel were brought through DG Cartesian to FP. In FP, this perception was linked to 'the curvature of the graphs'.

In the construction of the idea of acceleration, two of the pairs of students associated 'the variations of the speeds of x and y' and 'the fact that the sprite of y overtakes the one of x'. They identified sprites as having variable derivative when 'y overtakes x' while they were moving x in DG microworlds. Another pair of students only identified variable speed in the strikers when y disappeared quickly in the screen of DG microworlds.

Diana & Gisele and Jane & Anne developed perceptions in FP isolated from those built in both DG microworlds. Both sets of microworlds were linked through their previous knowledge of curvature. Diana & Gisele reached the link only in the final interview. The other two pairs worked linking all their perceptions which were derived from interactions with the microworlds.

Despite not being able to link acceleration to curvature in the pre-test, all the pairs of students used curvature as a property to compare parabolas from the pre-test. Nonetheless, they all exhibited difficulties in comparing curvature of parabolas with different turning points. The exploration of dynamic transformations of graphs in FP encouraged the students to measure curvature. Table AIV-5.3 and diagram AIV-5.1 show that the students revised the measure of curvature. The students tried to calculate the distance between two symmetrical points in a parabola, following a pointwise view. By exploring vertical translations, all the pairs of students, apart from Diana & Gisele, realised this was not a valid way to measure curvature. Moreover, they tried a method of measuring it using the idea of the vertical translation after realising that the curvature was invariant through this transformation (see table AIV-5.3). This method was scaffolded for two pairs of students, who began working with DG, as a way to compare curvatures of two graphs. They used the translation later as a way to compare the curvatures. In the case of Diana & Gisele, they went beyond a critical moment, when they could have seen that the previous idea did not work, but they only created a new rule for the case. This pair of students realised this incompatibility by stretching the graph of a sine vertically. In general, the stretch was used by the students to realise different curvatures of the graphs. Therefore, the use of dynamic transformations of graphs was important to the students' revision of their previous perceptions of curvature.

1.6 Range

The pre-test demonstrated a variety of previous perceptions of range. Apart from Bernard & Charles, all the pairs of students only thought of range as bounded and so associated with extreme values and turning points. Bernard & Charles only identified extreme values in pointwise graphs. In addition, two of the pairs only considered range as a discrete collection of outputs.

In DG Parallel all the students discriminated range as 'the place where y can move through' without naming it explicitly. Apart from Jane & Anne, all of them developed two approaches to analysing this idea: (a) dividing it into positive and negative — a polarised approach; and (b) considering the bounds of the motion of y. They also brought these two approaches to DG Cartesian.

As motion is a strong feature in DG microworlds, all the pairs of students discriminated the range of the strikers as 'the bound of the motion of y'. Only Jane & Anne did not explore this perception very far. The others generalised this perception to all bounded and boundless functions. Even the students who considered only bounded range in the pre-test generalised the perception to boundless functions in DG Parallel or in DG Cartesian. This sort of analysis was not generalised to constant functions which is further evidence of the importance of motion in this classification. Also, from a mathematical viewpoint the approach (b) considered range as a global set, not just discrete outputs. Unfortunately, it was isolated until the final interview.

The polarised approach to range was also exhibited by these three pairs of students. They divided the strikers into positive, negative, and positive and negative. Note that John & Tanya used this approach only for constant functions. Nonetheless, this approach persisted in the other pairs even in FP. Note that with this approach the other two pairs had difficulties in seeing the range of the striker of $y=0.25x^2-8$ as being similar to the range of the other strikers of parabolas.

Bernard & Charles and John & Tanya started using the polarised approach and with the development of the work in both DG microworlds abandoned it and moved into the approach (b). The other two pairs of students showed the same development but in parallel. Moreover, for these pairs the approach (b) was the one which lost importance.

Apart from Bernard & Charles, all the pairs of students exploited the motion of (x,y) to explore range in DG Cartesian. John & Tanya used the graph traced by (x,y) to generalise the idea of infinity to strikers where y did not disappear from the screen. For Diana & Gisele and Jane & Anne, the trace of (x,y) was linked to their previous knowledge of extreme values. Nonetheless, none of them used DG Cartesian as a bridge between DG Parallel and the Cartesian system.

All the pairs of students connected the perceptions of extreme values and 'the bound of the motion of y' only in the final interview. In the research environment these perceptions stayed completely isolated.

All the pairs, apart from Jane & Anne, linked their perceptions of range derived from interaction with FP to their previous knowledge of range or extreme values. Jane & Anne did not link them to any other perceptions developed in other microworlds or previous knowledge.

On exploring alterations of range in graphs provoked by stretches and translations in FP (see table AIV-6.3), all the pairs of students, apart from Anne & Jane, recognised and revised two aspects of range: the amplitude of the range and range as a set. Diana & Gisele recognised them but only in the final interview where they also identified the corresponding ideas in graphs. For the other two pairs, the transformations generated discussion about the real meaning of the term range. For Diana & Gisele and Jane & Anne, exploration of FP commands was responsible for overcoming the restriction of considering range only for functions with bounded range. In fact, on using the commands to try out their belief that range should be bound, they revised their perception of range generalising it to all the functions — bound or boundless. The interaction with dynamic transformations of graphs in FP enabled the students to overcome the limitations of perceiving range only for bounded functions.

1.7 Symmetry

In DG Parallel, all the students only perceived symmetry in terms of symmetric numbers. In the same way, in DG Cartesian the most common perception of symmetry also deals with symmetric numbers. Nonetheless, line symmetry was perceived only by Diana & Gisele using the shape traced by (x,y). In conclusion, line symmetry was perceived by the students only in microworlds which contain the graph (explicitly drawn) and one exception in DG Cartesian, which shows that line symmetry is usually perceived in a pictorial way having no special status in DG microworlds.

The students' perceptions of symmetry strongly emphasised symmetric numbers. Diagram AIV-7.1 shows that the majority of attempts to express line symmetry in pointwise or variational ways resulted in perceptions associated with symmetric numbers. This emphasis operated as an obstacle to the students' generalisation of these perceptions to line symmetry about a line different from the axes, which does not correspond to symmetric numbers. In their school mathematics, a pointwise correspondence for line symmetry is presented only for the graphs with line of symmetry in one of the axes which can be given in terms of symmetric numbers.

Although it was not easy to identify in DG Parallel, in the final interview all the pairs of students, excepting Jane & Anne, discriminated line symmetry within this microworld relating it to the motions of x and y — in a variational way. The association between line symmetry and symmetric numbers prevented one of these pairs from linking this variational perception to the pictorial one of line of symmetry, creating an obstacle for the students' connection of their perceptions while using the term symmetry. As for Anne & Jane, they had only discussed line symmetry involving symmetric numbers in the research environment, thus, in the final interview they connected line of symmetry in graphs to symmetric numbers in DG Parallel.

On exploring horizontal transformations of graphs, all the students, excepting Anne & Jane, generalised line symmetry for graph with line of symmetry different from the y-axis (see table AIV-7.3). John & Tanya and Bernard & Charles had a previous pictorial knowledge of symmetry constrained to line of symmetry in the axes. By generating examples with transformations of graphs in FP, these students realised that the line symmetry need not be about the axes. Charles & Bernard did this for line of symmetry different from the y-axis, and John & Tanya different from the x-axis. The other students were able to identify line symmetry on graphs with line of symmetry different from the axes by using the turning point as a way of recognising it.

All the students perceived line symmetry pictorially, while only some of them connected this perception to a pointwise perception through a correspondence with symmetric numbers. It is interesting that all the students were encouraged in FP to try to adapt a functional meaning for line symmetry. Diana & Gisele and John & Tanya used the generalisation of pictorial perception of line symmetry about a line different from the axes to generalise the pointwise meaning for line symmetry. Thus, they overcame the constraint of seeing line symmetry only as symmetric numbers. Note that among the students who generalised line symmetry about a line different from the axes in FP, only Bernard & Charles were unable to integrate the information to perceive the pointwise correspondence for this generalisation. Their attitude changed, however, when dealing with symmetric graphs, and they began to locate the lines of symmetry.

1.8 Periodicity

All the students presented pictorial perceptions of periodicity in the pre-test although none distinguished periodic from oscillatory graphs. In DG Parallel, John & Tanya were the only pair of students who developed a variational perception to periodicity by relating the behaviour of x and y and who were also trying to connect perceptions among different microworlds all the time. For Diana & Gisele and Jane & Anne, who explored the notion of periodicity in this microworld, the oscillation between 'y follows x' and 'y does not follow x' was the strongest perception of periodicity which stayed isolated. In the case of Anne & Jane, they only connected this perception by matching the strikers from both DG microworlds.

In DG Cartesian, unlike in DG Parallel, the students strengthened their perceptions of periodicity. All the pairs of students, apart from Jane & Anne, used DG Cartesian to connect their previous perception of period as 'the shape traced by (x,y)' to 'the one which relates x to (x,y) or x to y'. Moreover, after identifying periodic functions in DG Cartesian, Bernard & Charles, the pair who had not explored periodicity in DG Parallel, connected this idea back to DG Parallel in the final interview. Jane & Anne also connected their perceptions of periodicity from FP back to DG Parallel without using DG Cartesian. Nonetheless, they limited this connection to the periodicity of the roots. In conclusion, by exploring the contrast between 'absence of the shape' and 'motion of x, y and (x,y)' in DG Cartesian, three of the pairs of students developed a variational perception of periodicity. Thus, DG Cartesian composed a bridge from their previous knowledge of periodicity to its perception in DG Parallel. Note that in this process the students distinguished a periodic function from any oscillatory graph.

Table AIV-8.3 shows that all the pairs, apart from Diana & Gisele, exploited the dynamic transformations of graphs in FP to revise their previous perceptions of period. Jane & Anne did it in the final interview. Bernard & Charles and John & Tanya explored the transformations by generating counter-examples of previous perception which motivated them to revise these perceptions. For example, by perceiving the period as the interval between two roots and generating two graphs of sines with same period translated horizontally, John & Tanya concluded that the period was not 'the interval' but 'the distance' between the roots.

For all the students, periodicity remained a property discriminated using special points of graphs. Even Bernard & Charles who reached the invariance of the period when calculated on different points did this on special points.

2 Synthesis

The following analysis will discuss how students made connections between different perceptions of function properties, divided into two subsections: spontaneous synthesis where the students made connections while interacting in a microworld and motivated synthesis where connections were motivated by the researcher in the final interview.

2.1 Spontaneous synthesis

The connections spontaneously made will be analysed according to the microworld in which they were made.

Connections in DG Parallel

Table AV-2.1 shows that the pairs of students who began working with DG Parallel made very few spontaneous links between perceptions built in DG Parallel and previous knowledge. For instance, the only idea from previous knowledge used by Bernard & Charles was infinity which they used to explain what happened when a 'striker gets out of the visual screen'. The majority of the perceptions constructed in DG Parallel by these students were discriminated and generalised within this microworld (see table AV-2.2). The other two pairs of students more often built connections using their knowledge of the graphs explored in FP as well as bringing terms from previous knowledge to make sense in DG Parallel (see table AV-2.1). These results show that matching the graphs and strikers was important in encouraging the students to make connections while working with DG Parallel.

Synthesis in DG Cartesian

While exploring DG Cartesian, the students characterised the strikers in two ways: (a) by the shape formed by the motion of (x,y) which was usually linked to perceptions derived from both explorations in FP and previous knowledge and (b) by comparing the motions or the values of x and y, usually linked with perceptions derived from interaction with DG Parallel. Table AV-2.2 shows that the pairs of students varied in presenting one way or both ways for different properties. Moreover, in most of the cases when the students characterised the strikers in both ways, they developed these independently of each other in DG Cartesian. Many of them were linked in FP or in the final interview. Bernard & Charles illustrated this when they connected the idea of speed from DG Parallel to DG Cartesian without linking it to 'slope of graphs'. As table AV-2.2 shows, these links mainly occurred in DG Cartesian for turning points and constant functions. The pairs of students who finished by working with FP also linked the two perceptions of derivative and second derivative while exploring FP. Thus, DG Cartesian worked as a bridge from perceptions articulated in DG Parallel to DG Cartesian mainly for all properties linked with variation and for turning point. It is interesting to notice that monotonicity was the only perception of variation in which DG Cartesian was not explored as bridge. For this property as well as for symmetry and periodicity, the majority of students concentrated on analysing the shape and were blocked from building connections with previous knowledge. Note that the majority of these perceptions were previously articulated pictorially.

Table AV-2.2 also shows that the connections were more often articulated with previous knowledge or the graphs in FP than with DG Parallel. The motivations labelled 'A' and 'D' in this table show that the students were often prompted to connect perceptions derived from different microworlds in DG Cartesian by recognising the family of functions to which the strikers belong and by using terms already studied for this family. For example, Bernard & Charles brought the term periodicity to make sense in DG Cartesian after remembering the trigonometric functions, which they called "up and down".

The possibility of looking at the behaviour of x and y and at the trace of the graph simultaneously and separately was the main reason why the students used DG Cartesian to bridge variational perceptions from DG Parallel to the Cartesian system (see motivations C and F in table AV-2.2). This possibility allowed two kinds of connection: (a) the use of perceptions constructed in DG Parallel to understand properties in Cartesian system and (b) the use of shape to make sense of previous perceptions by comparing the behaviour of x and y. In each case, DG Cartesian was used by the students as a bridge between variational and pointwise perceptions (built in DG Parallel or in DG Cartesian) and pictorial perceptions in the Cartesian system. The case (a) can be illustrated by John & Tanya's connection between constant function and 'y is independent of x'; this pair of students had recognised the family of the strikers from DG Parallel. The possibility of manipulating x and seeing y in DG Parallel enabled them to perceive the constant functions as 'y is independent of x' linked to 'horizontal straight line'. Tanya argued that 'x moves, moves, but y does not move'. The case (b) can be illustrated by the fact that after recognising the shape of the striker given by $y=7\sin(0.25\pi x)$, Diana & Gisele started investigating line symmetry. At this point, they tried to say what line symmetry means for the sprites representing x, y and (x,y). The kind of bridge (b) shows that the use of DG Cartesian encouraged all the pairs of students to search for a variational or pointwise correspondence for properties of which they had a pictorial perception, although sometimes they did not reach a mathematically correct connection. For instance, in the example above, Diana & Gisele did not reach the corresponding idea of symmetry, they reached a periodic aspect instead.

Synthesis in FP with previous knowledge

According to table AV-2.3, while exploring FP, all the pairs of students worked by connecting perceptions of all the properties with their previous knowledge. This table summarises the moments at which the pairs of students were motivated to connect their perceptions. One of the most frequent moments was when the students brought terms from their previous knowledge to make sense in FP, which motivated discussion about their meaning. The term 'period' for example was one that all the pairs of students knew. John & Tanya talked about period in two senses: 'repetition of a trajectory' in graph, and 'interval of x after which its trajectory repeated'. While stretching vertically the graph of $y=7\sin(0.25\pi x)$, they realised that 'trajectories' could be completely different without altering their interval. Later, translating horizontally the graph of $y=7\sin(0.25\pi x)$, they realised that what was important was the length of the interval. At this point, they calculated the period. This example also illustrates another moment which appeared many times as motivating connections (see table AV-2.3). The students analysed the properties as variants and invariants of the transformations of graphs.

The students were motivated to build connections in three other situations. One of them was when they attempted to distinguish their descriptions of two or more functions. This situation shows the importance of the nature of the designed activities in leading students into connections. The other two are linked to the nature of the activities while interacting with dynamic transformations of graphs. The use of both algebraic and Cartesian representations while transforming graphs helped all the pairs of students to link different characteristics in different representations. After tracing the graph of $y=6$ and trying to transform the graph of $y=2x$ into it, for example, Tanya made sense of the equation $y=6$ as "y has only one value while x can have many values". The last motivation was a consequence of the students' attempts to make sense of results obtained from transformations which are counter-examples of their own beliefs (see table AV-2.3). This motivation is interesting because it emphasises a difference between FP and DG microworlds. In FP one can generate examples and counter-examples while in DG microworlds the examples are given. Note that in table AV-2.2 counter-examples were used twice to make connections, in one case in FP by generating counter-examples.

General points

Label 'NL' in tables AV-2.1, AV-2.2, AV-2.3 and AV-2.4 points to connections that the students made linking perceptions with 'special status'. For example Bernard & Charles and Diana & Gisele connected 'turning point' to 'point where y meets x', which is a connection between properties perceived as 'special points'. John & Tanya presented a connection between 'angle formed between straight line and the x-axis' and 'an imaginary angle in DG Parallel' which is also a connection between same object 'angle'. Connections of this kind also appeared by linking the 'adjectives'. For example, Bernard & Charles connected 'positive angular coefficient' in quadratic equations to 'positive' range in parabolas. Nonetheless, no synthesis of this kind was significant among the connections made by the pairs of students. Moreover, the majority of them were revised by the students. However, these kinds of connections appeared more as associations which will be discussed in section 3.

2.2 Motivated synthesis

The two activities of the final interview were crucial for the students to synthesise their perceptions between different microworlds: (A1) matching the strikers and the graphs and (A2) guessing the change in a striker after transforming its corresponding graph in FP (see table AV-2.4). Many of the connections were also provoked only with direct questions.

According to table AV-2.4, activity A1 led all the pairs of students to connect perceptions from different microworlds as well as to revise and generalise some perceptions using the connections (see 'GP' in the table). Bernard & Charles, for example, revised their link between 'inclination of graph' and 'ratio between x and y' to 'ratio between the variations of x and y' after matching the graphs and strikers of y=x and y=x-6.

Activity A2 led all the students to make connections by searching for a new perception in DG Parallel (see table AV-2.4). The generation of examples and counter-examples encouraged the students to search for a meaning in DG Parallel for their perceptions in graphs. Observe that this activity provoked the students to connect perceptions mainly for the properties of range, symmetry and periodicity. Remember that these properties were not thoroughly explored in DG Parallel.

As shown in table AV-2.4, the final interview was useful for the students:

- to generalise perceptions of properties which were previously restricted to one family of functions (see 'GP' in the table);
- to search for the perceptions in DG Parallel brought from previous knowledge of graphs or from interactions with FP. Columns 'DG Parallel' and 'Graphs and definitions' in the table show that the majority of connections were built in the final interview by searching for a new perception in DG Parallel which would correspond to the one spontaneously expressed in Cartesian system. All the students started explaining 'the shape of graphs' by 'the behaviour of x and y'. This finding demonstrated that the work with DG Parallel and DG Cartesian was useful in giving the students a variational analysis of graphs. This helped in promoting bridges from Cartesian System to DG Parallel;

- to connect corresponding perceptions which remained isolated in different microworlds. For example, the pairs of students connected the perceptions of all the properties, apart from those of symmetry, which stayed isolated (see table AV-2.4). In the case of symmetry, it should be noted that almost no perception was built in DG Parallel.

3 Associations

In the development of the students' perceptions of the properties, the analysis shows that the students spontaneously constructed and revised associations between different properties. Table AV-3.1 shows the leaps taken by the students when revising the associations and the cases when associations were not revised. Column 'Origins' in the table investigates causes of the associations as well as patterns presented in the associations. It can be divided into four categories: those constructed in the research environment (A to C); those which reflect a tendency in students' perceptions (E to G); those which have similarities with school curriculum (H to K). Origins D will be discussed in section 5 while discussing the role of the microworlds.

As table AV-3.1 shows, this categorisation is not exclusive, for example, there are associations with origins in the research environment and also with similarities with school curriculum such as the association between 'periodic function' and 'oscillatory graphs' presented by all the pairs.

3.1 Origins in the research environment

A legitimate way of recognising a property

Table AV-3.1 shows that mainly for the properties of monotonicity, derivative, second derivative and periodicity, associations were developed as a legitimate way of recognising a property among a limited group of functions. Nonetheless, only for the properties of variation were these associations clearly separated from similarities with school curriculum. The fact that all the pairs of students started to construct the idea of derivative by associating it to 'ratio between absolute values of x and y', instead of considering the variations of x and y illustrates this sort of association. It was recognised and revised by all the pairs of students while they were analysing the striker of y=x-6.

The building of the associations seems once more to indicate a natural process in the construction of knowledge. Yet, what is really interesting is that almost all these above-mentioned associations (see 'A' in table AV-3.1) which had no clear similarity with the school curriculum, were later revised by the students while analysing counter-examples.

Other reasons

Less frequently, two other causes of associations were detected as originating in the research environment. Associations with origin 'C' in table AV-3.1 were built while linking perceptions from different microworlds such as 'y is motionless' and '(x,y) is motionless' for two of the pairs of students. Origin 'B' in table AV-3.1 shows that comparison between invariant properties while transforming graphs was the reason for the building of associations. For example, two of the pairs associated and did not revise 'inclination of straight line' and 'ratio between x-intercept and y-intercept'. Nonetheless, table AV-3.1 shows that these associations were not so frequent as the use of these transformations to revise associations (see Revision [GC]).

3.2 Similarities with the school curriculum

Properties studied only for a particular set of functions

Table AV-3.1 shows many associations with origins as a legitimate way of recognising a property among one family of functions or a set of functions within a family (see origin I) — the emphasis of the school curriculum. These associations were frequent with turning points, monotonicity, extreme values, line symmetry and periodicity. Note that in the case of periodicity, these associations also provide a legitimate way of recognising the property among the twelve functions selected (see origin A). An illustration of these associations is that the school emphasis on dealing with increasing for linear functions led all the pairs of students to associate 'increasing' to rules involving positive and negative — 'the side where the straight line is positive' (see table AV-3.1).

Associations linked with use of terms

This association was interesting because it only appeared when the students were using the term 'increasing' for monotonicity. For example, when John & Tanya created another term for the same characteristic identified by the behaviour of x and y, this association no longer appeared. This is also used as evidence as to the origins in previous knowledge. By using mathematical terms, the students restricted perceptions to some cases of the properties. This use is also observed in the association between 'line symmetry' and symmetric numbers which three of the pairs presented and that also appear in the school curriculum analysis (see chapter VI).

Special points

Another sort of association derived from school emphasised special points (see origin J in table AV-3.1). The perceptions most affected in this way were those of turning point, periodicity and extreme values. Those are also the properties which the school emphasises by special points in graphs. For example, all the pairs of students limited their perceptions of periodicity to that of special points in special 'periodic roots'. This tendency is also observed in the analysis of the curriculum.

Over-generalisation of the role of coefficient

Finally, table AV-3.1 shows one more association with origins in the school curriculum which appeared only twice in DG Cartesian but has interesting origins (see origins H). This association originated from an over-generalisation of the role of the coefficient 'a', in a general formula from $y=ax+b$ to $y=ax^2+bx+c$. This association was more clear while Diana & Gisele were working with the graph of $y=0.25x^2-8$. They linked increasing function to positive curvature using the fact that 'a' is positive. Note that as in school mathematics it is usual to use the general formula for linear function as $y=ax+b$ and the general formula for quadratic functions as $y=ax^2+bx+c$, their association seemed to be natural. Natural because the students learnt that a linear function given by '$y=ax$' is increasing if 'a' is positive, but they also studied quadratic functions denoting the angular coefficient by 'a'. It is reasonable that they think 'a' plays the same role in the quadratic equation. The association was also caused by the fact that the idea of monotonicity was not much emphasised in the family of parabolic functions at school.

3.3 Patterns in associations

Many associations were made on the following basis: (a) a tendency to interpret properties (especially those linked with variation) in a pointwise way, (b) a tendency to transform a property into a rule involving polarisation and (c) a tendency to use the same object or same adjective as a reason to associate properties.

Pointwise view of functions

The tendency (a) was exhibited by all the pairs of students while exploring derivative, second derivative and periodicity. The association between 'being quicker' and 'being ahead of the others' which was expressed by all the pairs of students is an illustration of this tendency. Note that all these associations, apart from those linked with periodicity, were revised by the students. Also all the pairs of students exhibited associations which reflected the pointwise view of functions when trying a functional meaning of their pictorial perceptions of periodicity and second derivative. For example, on trying to find out the functional meaning for curvature, all the pairs of students associated it to 'the distance between two symmetrical points'.

Polarisation of Knowledge

Tendency (b) was exhibited while the students where analysing the properties of monotonicity, range and periodicity (see origin F in table AV-3.1). For example, all the pairs of students associated the term 'increasing' to 'straight line which is positive in the positive side', which is also a rule predicted in the analysis of the school curriculum (see chapter VI). This tendency was also reflected in the fact that the students divided the domain into positive and negative to analyse any property, causing associations such as the one between 'y is oscillatory' and 'y is independent of x'. Note that for only two of the cases the associations were not revised, but half of the revisions were done in the final interview.

Association using the same object

Tendency C was mainly exhibited by all the pairs of students while analysing turning points, constant functions and range (see origin E in table AV-3.1). An illustration of association by same object can be seen by 'inclination of straight line' associated with 'imaginary angle' by John & Tanya in DG Parallel. Properties were also associated because they are characterised with the same adjective, for example, positive range and positive angular coefficient. Nonetheless, apart from two cases, all these associations were initial and temporary, almost all easily revised by the pairs of students.

3.4 Revision of associations

Counter-examples generating critical moments

Although the process of revising associations was very particular to each individual and could not be characterised by properties, table AV-3.1 shows that generally revisions happened in critical moments and most notably often interaction with counter-examples (see revision [T] and [GC] in column revision). Together both cases composed the majority of revisions of associations made by the students. [T] represents the moments when the students tried to generalise an association to a different function which represents a counter-example of the association. For example, as mentioned before, the analysis of the association between 'speed of striker' and 'ratio between absolute values of x and y' in the striker of y=x-6 was for all the pairs responsible for revision of the association. [GC] shows a special kind of [T] when the counter-examples were generated by transformations of graphs. Diana & Gisele, for example, revised the association between 'curvature' and 'distance between two symmetrical points' while stretching vertically the graph of $y=7\sin(0.125\pi x)$.

Nonetheless, as table AV-3.1 shows, the associations were not revised every time the students passed through critical moments such as examining counter-examples (see [WCM] in table AV-3.1).

Absence of critical moments

Table AV-3.1 shows some cases in which the associations were not revised but nor did the students pass through any critical moments such as examining a counter-example (see Revision [NCM]), showing once more the importance of interactions with counter-examples in revising associations. Some of the associations did not have counter-examples in the research environment, in other cases the students did not examine the counter-examples. The association between 'periodic function' and 'oscillatory graph' is an illustration of the case in which the research environment does not present counter-examples. For these cases, I am not sure if the presence of counter-examples would help the students to revise the association or whether the association would hinder the distinction between periodicity and oscillation.

4 Obstacles

The students were prevented from generalising, linking perceptions and perceiving similarities of functions or even investigating new ideas by their previous perceptions. Patterns of similarities were identified in these perceptions (see table AV-3.2). Below, I discuss some of these patterns and the obstacles they caused in the students' development of perceptions.

Pointwise perceptions

The interactions with the microworlds led the students to change their initial tendency to analyse the function properties in a pointwise way, mainly for turning points and variation. As soon as he was informed that the strikers in DG Parallel represented functions, John stated: "it seems to be more difficult ... because [in] this [activity], we have to think of it [striker] as function, think of it not only as a game, it is not only [to think] in their [strikers'] motions, but there are other items that I think will also appear". This statement reflects the separation presented by all the pairs between motion as a property of strikers and pointwise perceptions as a property of functions (see the associations in table AV-3.1). This separation had been preventing the students from building generalisations and connections. Nonetheless, for all the properties linked with variation, these barriers were transposed, which indicates that the interaction with the dynamic microworlds helped the students to overcome barriers derived from a pointwise view of functions while investigating properties of variation. On the other hand, for other properties such as linearity and periodicity, the barriers were not easily overcome. In the case of linearity, for example, concepts were transformed removing their original sense to a rule of recognition almost completely based on discrete points. Diana & Gisele used 'graph passes through (0,0)' replacing the meaning of linear function (see table AV-3.1). In the case of periodicity the emphasis on special points prevented the students from having a global perception of periodicity among all the points of the domain.

Note that for these two above-mentioned cases the school emphasis on special points coincides with these tendencies, thus increasing the difficulty in overcoming them. Breaking a tendency seems to be easier than revising a knowledge 'well established' by the students in the school curriculum.

Tendency to polarise knowledge

The tendency to polarise mathematical knowledge appeared also as an obstacle to generalising properties as well as to linking properties between microworlds. Table AV-3.2 together with the associations generated by the tendency towards polarisation show that this kind of obstacle was stronger than the pointwise one. For example, the tendency to divide a set into positive and negative prevented all the pairs from recognising similarities in range or in monotonicity among functions of the same family. Note that in the case of range, the approach which involved limits of motions led three of the pairs to move from the polarised approach into a topological one in DG microworlds. This enabled them to transpose the obstacles. Thus, the interaction with the dynamic aspect of DG microworlds was responsible for changes in these polarised approaches.

Also, tables AV-3.1 and AV-3.2 show that on dealing with mathematical terminology, the associations and obstacles became more difficult to overcome. For example, the obstacle concerning the use of the term 'increasing' was overcome for the majority in the final interview, after marked generalisation of other perceptions of monotonicity.

Emphasis on some properties for some families

The emphasis which school mathematics placed on some properties in a particular family of functions seemed to create obstacles to the students' development of perceptions of the properties as shown by table AV-3.2. After recognising the family of functions that each striker belonged to, the students completely changed their approach to analysing the strikers, reacting in a way which blocked progress to the next step. First, the students were led into associations which distorted the original meaning of some concepts (see table AV-3.2). Second, they sought only for properties emphasised at school or they stopped searching for new characteristics (see table AV-3.2). John & Tanya and Diana & Gisele did both. Since they did not analyse a property which they had not been taught in the family, they were prevented from generalising or revising their perceptions.

Equation as essence of function

In contrast with the results obtained in the pilot study, in the main study the interference caused by the consideration of equation as being the 'essence of a function' appeared only twice (see table AV-3.2). It seems that the change promoted in the methodology of the study led the students to focus more attention on graphs than on equations.

Specifying the variables

I had many opportunities to observe that all the pairs of students characterised the functions without specifying the subject that they were talking about (see table AV-3.2). For example, "it is positive" without mentioning what 'it' means. This imprecise language caused associations of properties and a failure to separate the variables. In the activities, which involved describing/guessing, one of the partners always asked the other to be specific in what s/he said. Also, more precise language was needed in using DG microworlds, since it was not a familiar representation to the students. I argue also that in DG microworlds, the clear separation of the objects x and y also made them more precise in their language. For example, despite not recognising maximum and minimum in the bound of the motion of the striker, Diana & Gisele clearly localised the limit in y, not in x.

5 The role of the microworlds

This section will discuss some common points in students' perceptions while interacting with the microworlds.

Lens for amplifying associations

The observation of the students' interactions with the microworlds served as a 'lens' (Hillel et al, 1992) on the associations students made as well as the reasons which provoked them (see table AV-3.1). For a better understanding of the metaphor, I will illustrate with the case of the association between 'Parabola' and 'curve with turning point' presented by all the pairs in at least one of the microworlds presenting the Cartesian representation. For two pairs this was also presented in DG Parallel, a microworld where shapes of graph were not available, and in which this association was more evident. Before identifying the idea of curvature in DG Parallel, John & Tanya used turning point to recognise parabolas, even for the striker of $y=7\sin(0.25\pi x)$. John also described how to distinguish the curves from the straight lines "what makes it become a curve is it [y] arrives to a point and returns". Another illustration concerns the interaction with FP, when all the pairs of students associated 'curvature' to 'distance between two symmetrical points'.

5.1 DG microworlds

> **John:** It is interesting... When we stop to think, we see only functions, only looking at the game [DG Parallel].
> **Researcher:** Really?
> **John:** That's incredible!
> **Researcher:** Is it? You see the functions in the strikers?
> **John:** Yes... its motion. It is interesting the motion of the functions, just in a game like this, we had never imagined, it is as if the game masks...
> **Researcher:** Is it hiding...?
> **John:** I remembered ... I was comparing to something... to the money-lender. Today, money-lending is illegal, isn't it?
> **Researcher:** Yes.
> **John:** Once my father went to a money-lender, the money-lender was in a clothes and shoes shop, when we arrived at that shop there were the sellers.
> **Researcher:** Hum...Hum.
> **John:** But, when we went into [the shop], he [his father] said: I came to give you money. So, you could take the lift to go to the money-lender. This game is similar, this is, it seems to be so simple but the truth is that it shows you more about complex functions, and shows you the motion of these functions, their relations...
> **Researcher:** Yes, it is like the behaviour of the functions.
> **John:** Exactly.
> ....
> **John:** It should be very useful in a school. For example, I started to understand how valuable functions are with this work.

The above transcription was a special moment when one of the students clearly stated the usefulness of the interaction with DG Parallel. He argued that this microworld stressed motion in the function. From my viewpoint, he was able to perceive that the same idea can be seen in different representations, as well as to observe that each representation emphasises different characteristics.

Concentrating on variational views

The interactions with DG microworlds led the students to concentrate on variational perceptions mainly for the properties of turning points and variation (see tables AV-4.1 and AV-4.2). They also approached the property of range only by considering the motion of y while looking at its limit. Nonetheless, the interaction scaffolded a variational view of graphs mainly for turning points, constant function and derivative (see codes C in the tables). Thus, the use of DG microworlds scaffolded a new way of analysing the graphs. This was demonstrated in the use by Charles & Bernard, who began by working with DG, of the same method of exploration of DG Parallel to verify the variation of graphs in FP.

Search for functional meaning of pictorial perceptions

DG Cartesian microworld encouraged the students to search for a functional correspondence to pictorial perceptions (see code B in tables AV-4.1 and AV-4.2). I argued that the contrast between the possibility of seeing the shape of the graph by the motion of (x,y) and the absence of its trace was crucial for the change in these students. This remark is based on the fact that these searches always happened after a student brought a view from their previous knowledge to characterise a striker in this microworld. Nonetheless, without the drawing of the shape when the other students came to guess the striker, the first student tried an explanation using the behaviour of x and y.

Separation of variables

It is also important to emphasise that the interaction with DG microworlds encouraged all the students to define which object they were talking about while describing the functions (see code A in tables AV-4.1 and AV-4.2). This was reflected in the fact that the students started identifying the variable they were talking about. This was more apparent in relation to the properties related to variation. For example, comparing Diana & Gisele's arguments about periodicity, we can notice a difference. In FP, they argued that it repeated in the graph. As for DG Cartesian, "the point repeats its path,... each 4 units x moves, y makes one turn".

5.2 The interaction with dynamic transformations of graphs (FP)

The interaction with dynamic transformations of graphs led the students to revise and generalise their perceptions of the properties. John's observations after exploring the commands in the graphs of y=abs(x) provided a special moment showing that in fact these transformations really interfered with students' perceptions. John complained about the possibility of Tanya using the commands when trying to guess the function described by him saying: "It will be very easy because the commands give you some hints". This led me to have a closer look at the 'hints' revealed by the commands.

Generating their path of learning

First, as table AV-3.1 shows, while interacting with transformations of graphs the students generated examples of and counter-examples to their own perceptions, hypotheses which led them to revise the associations. Thus, the students followed their own path of learning while interacting with FP. Nonetheless, on a smaller scale, the exploration of transformations of graphs also led them into associations. Also table AV-1.7 shows that all the pairs, apart from Jane & Anne, used these commands to revise the association between two properties and distinguish them in the research environment.

Table AV-1.7 shows that while transforming graphs or trying to make sense of results obtained after these transformations (see code 1), the students generated examples and counter-examples of ideas, discovered new perceptions of a property and discovered important aspects of a known perception. All this happened mainly for the properties of turning point, constant function, range and symmetry, (and also once for periodicity and second derivative).

Search for functional meaning of pictorial perceptions and separation of variables

It is interesting that only Jane & Anne used the interaction with the transformations to separate the variables x, y and (x,y). Nonetheless, all the pairs of students were led into a functional search for properties pictorially perceived (see code 8 in table AV-1.7). The search also led them to separate the variables.

Overcoming limits of compartmentalisation of knowledge

An interesting result obtained by exploring the transformations, although it only occurred twice, was that two of the pairs were able to overcome the limitations imposed by the way the school mathematics compartmentalised knowledge (see code

11 in table AV-1.7). While transforming graphs, the students observed relationships between different properties.

The role of the dynamic transformations

Table AV-1.7 highlights the importance of the dynamic nature of the transformations of graphs in enabling the students to connect perceptions by observing variants and invariants of the transformations (see code 4). Moreover, a general overview of table AV-1.7 shows that the most frequently explored transformations were translations and stretches. In the case of reflections the only dynamic transformation was in the choice of the mirror line. This suggests that once more dynamic transformations played a important role for the progression of the students' perceptions. Also note that among all transformations the most often explored were the vertical ones.

The translations and stretches were also explored to generalise perceptions among different functions. These generalisations involved revision of perceptions in order to apply to qualitatively different functions (see code 6 in table AV-1.7).

Development of measures

An interesting result of the study was that all the pairs of students used the vertical translations to develop a measure for derivative and three of them did this for second derivative. This was a measure for what was generally pictorially perceived (see code 7 in table AV-1.7).

Concentrating on graphs and scaffolding a new way of sketching graphs

The interaction with the transformations of graphs scaffolded a new way of sketching graphs for the students as well as allowing them to switch their attention from equation to graphs. The two pairs of students who began by working with FP tried to match the strikers with graphs while the other pairs used equations. In the case of John & Tanya, the use of FP commands scaffolded a new way to trace the graphs. Still working in FP, they tried to guess at $y=0.25x^2-8$ by imagining translations and reflections in the graph of $y=-0.25x^2$. Also Diana & Gisele sketched the graphs corresponding to the strikers.

Linking Cartesian to algebraic representations

The students did not base their observations while transforming graphs on linking algebraic to Cartesian representations in Graph window of FP. Note that code 5 in table AV-1.7 represents all the work done between the two representations while the others refer to explorations of perceptions within graphs. Considering that the methodology of the study was reformulated to ensure that the students concentrated on graphs rather than examining equations, this result was expected. Nonetheless, the cases when the link was observed show that the dynamic modifications of graphs followed by the corresponding algebraic modifications helped the pairs to link their perceptions of algebraic and Cartesian representations.

IX — Discussion of the Results

This chapter will discuss the main issues arising from the empirical study in relation to the previous results.

1 The properties as represented in the different microworlds

This section will consider each of the properties in turn, in the light of the relevant literature.

Turning points

In DG Parallel all the pairs of students perceived turning points in a variational way which differed from their previous perceptions in graphs. The pairs of students who began by working with FP also spontaneously brought this variational perception to graphs by exploring DG Cartesian as a bridge which links DG Parallel to the Cartesian system. The other pairs brought this variational view of turning points to graphs only in the final interview. The previous perceptions of my students coincides with the ones reported by Confrey (1992a) and Goldenberg (1988). Turning points were perceived by the students as special points in the Cartesian system. Thus, my students' view of turning points evolved from a pointwise to a variational one.

Goldenberg et al (1992) reported that their students returned to a pointwise view of the property when scales were introduced in DynaGraph. My results differ from theirs because although DG Parallel and DG Cartesian presented scales, all the pairs of students used the research environment to connect this variational perception to their knowledge of graphs. The nature of the describing and guessing activity was also one factor responsible for these different results as qualitative properties enabled the partner to guess the function described.

Two of the pairs also used 'special points' to connect their perceptions of turning point from graphs to DG Parallel. This focus on special points was an initial, temporary and naive connection. In contrast with the results of Moschkovich (1992), only half of the pairs of students used 'special points' to link their perceptions of turning points between DG Parallel and the Cartesian system.

In FP, all of the pairs explored transformations of graphs to generate examples and counter-examples of their own perceptions of turning points, but only two of them used these transformations to revise their previous perceptions, producing new ones.

Constant function

While describing and guessing functions in DG Parallel, all the students clearly distinguished linear from constant functions. The direct manipulation of x with feedback of the variation of y enabled all the pairs to perceive constant function as 'y is motionless' and 'y is independent of x'. These two perceptions were connected by all the pairs of students with 'horizontal straight lines'. Three of the pairs of students used DG Cartesian as a bridge to connect these variational perceptions with their previous knowledge.

The explorations of DG microworlds[1] led the students to question their previous representation of 'a motionless behaviour' as a dot in a graph. This behaviour was reported by Mevarech & Kramarsky (1993) and Goldenberg (1988) while working in other media. The interaction with dynamic representation of DG microworlds scaffolded a variational way of interpreting a graph leading to a perception that horizontal straight line is due to the fact that 'y is constant'. These results confirm the suggestion of Goldenberg et al (1992) that the use of the sequence DG Parallel to DG Cartesian can serve as a bridge for the construction of a variational analysis of the Cartesian system for constant functions.

As regards FP, it was used by all the pairs of students to explore the algebraic representation of constant function, by linking it to the Cartesian one. Bakar & Tall (1991) reported that their students had difficulty in considering equations of constant functions as representing functions. In contrast, all my students developed their perceptions by linking 'absence of x' in the equation through 'y is independent of x' to 'horizontal straight line'. In this case, the equation did not have to be changed to foster this development. In fact, it was achieved by 'absence of x' in the equation, instead of changing the appearance of the equation as in Bakar & Tall (op.cit.).

Also two of the pairs used dynamic transformations of graphs in FP to overcome the compartmentalisation of constant, monotonicity and derivative.

Monotonicity

In DG Parallel all the pairs of students perceived monotonicity in a variational way — as 'y follows x' or 'y does not follow x' — for all the functions, as opposed to their

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[1] By DG microworlds I mean the sequence of DG Parallel to DG Cartesian.

previous restriction to linear ones. Two of them also used these perceptions to generalise monotonicity to non-linear graphs, the others were blocked in this generalisation by the restrictions of studying monotonicity only in linear functions in school mathematics. The use of the term 'increasing' remained limited to linear functions for two of the pairs of students. In DG Cartesian, 'the direction of graphs' and 'orientation of the motions of y and x' were used to discriminate monotonicity, but only one pair of students recognised both perceptions as being one property.

The idea of monotonicity when analysed in DG microworlds differed substantially from the problems reported by Hillel et al (1992). My students developed the perception 'y follows x' in these microworlds by isolating each variable to which they were referring. Thus, the describing and guessing interactions with dynamic aspects of the microworlds allowed the students to consider x and y as variables when dealing with the idea of monotonicity.

In FP, three pairs of students exploited the transformations in diverse ways obtaining new aspects of monotonicity in graphs and introducing terms to discuss. One of these pairs overcame the compartmentalisation between monotonicity and derivative identified in their school knowledge as well as generalising the perception to other families such as parabolas. Another pair generalised a variational view to non-linear functions. The other brought the term 'increasing' into the discussion.

Derivative

In DG Parallel all the pairs of students reached a variational perception of derivative — 'comparing the speeds of x and y' — and generalised it to all the functions. Moreover, they all connected this perception to 'inclination of straight line'. The students started by a pointwise correspondence for this idea in which the comparison was calculated by a ratio between 'absolute values of x and y' which is comparable to the association reported by Clement (1985) 'height for slope'. The exploration of the striker of y=x-6 led to a critical moment for changing their perception to the variational one which 'compared the variations of x and y'. It also led them to connect this variational view to 'inclination of straight lines'. This result once more shows the limitations of the analysis of 'misconceptions'. Here, a pointwise correspondence was the starting point for a variational one.

In FP, there was a pattern between the perceptions of derivative the students constructed and the transformations explored. The invariance of 'slope' and of 'ratio between y-intercept and x-intercept' while translating straight lines led three of the pairs to identify these two properties as being the same. The explorations of stretches of linear functions led all the pairs to search for a functional meaning for 'the inclination of the straight lines' as 'ratio between x and y'. In the same way as reported by Confrey et al (1991b) while exploring tables and graphs in FP, my students built a variational way of analysing derivative in the research environment while exploring only the Graph window of FP.

Second derivative

By exploring DG microworlds all my students distinguished between 'constant and variable derivative', also linking it to 'curve or straight line'. Three of the pairs reached the distinction and link when describing and guessing the strikers in DG Cartesian. The last pair achieved the connection when motivated by the final interview. These results coincide with the ones reported by Confrey et al (1991b) while their students were exploiting functions in tables and graphs of FP. In my case, three of the pairs associated the idea of 'variable derivative' with views such as 'y overtakes x' or 'y leaves the screen speeding up' which were valid for the examples used by the students to build the distinction. Thus, a question remains here: will these perceptions form a barrier for students in later studies? If so, can it be overcome? These findings show once more that students' perceptions cannot be analysed from a purely negative aspect. In fact, these perceptions led the students to create the division between constant and variable derivative but the perceptions were not free of limitations or correct from a mathematical viewpoint. This evolution also shows the importance of letting students articulate perceptions within a microworld.

Despite reaching the distinction between 'constant and variable derivative' linked with 'curve or straight line', the students did not observe the constant second derivative of quadratic functions in any of the microworlds. This contrasts with the result obtained by Confrey (1992a). For example, they did not distinguish the variations of speed from quadratic to sine functions.

In FP, by exploring translations, three of the pairs of students revised their perceptions of curvature while comparing two parabolas. Moreover, they used the command for a comparative measurement of curvature of parabolas, realising also that 'the distance between two symmetrical points' does not measure the curvature. According to Goldenberg (1988) students usually present a visual illusion while comparing two parabolas which differ by a vertical translation. The exploration of dynamic transformations of graphs led my students to notice the 'visual illusion'. A similar result is reported by Borba (1993), but the results of the present study go further. It shows that the students also tried a way to measure curvature and realised the unfeasibility of measuring curvature by taking two symmetrical points. This process also allowed me to verify the way the students measured curvature serving as a 'lens' under which their perceptions became observable.

Range

The topological way of perceiving properties in DG Parallel reported by Goldenberg et al (1992) helped half of my students to change from a polarised approach to range to one which considers limits of motion. This new approach allowed the students to compare the range of different parabolas as being similar and different from that of linear functions. In DG Parallel, all the students perceived range as 'the place where y can move' adopting two focuses: dividing into positive and negative and considering 'bound of the motion of y'. These two focuses were also brought to DG Cartesian. They all generalised the range to boundless functions, breaking the previous limits of applicability of range. Two of the pairs easily abandoned the focus on polarised ideas in favour of a focus on 'limit'. Only in the final interview, did all the students connect 'bound of the motion of y' to extreme values in graphs.

While interacting with transformations of graphs in FP, the students generalised previous ideas of range to bounded and boundless graphs and revised previous perceptions discovering diverse aspects of range. Two of the pairs overcame the limitation of looking at range only for bounded graphs which is the emphasis in school. Almost all the pairs distinguished amplitude of range from range as a set. One of them realised the difference between turning point and maximum.

Line symmetry

Line symmetry was perceived by the students only in microworlds which contain graphs (explicitly drawing) with one exception in DG Cartesian. This shows that line symmetry is usually perceived in a pictorial way so it has no 'special status' in DG microworlds[2]. The majority of attempts to express line symmetry in a functional way resulted in perceptions associated with symmetric numbers as emphasised in school. In the final interview, three of the pairs searched for a functional meaning for their pictorial perception of line symmetry.

Almost all the pairs explored FP to generalise the idea of line symmetry for functions to a line of symmetry different from the y-axis. Nonetheless, when searching for a functional meaning for this new line of symmetry, they were not able to generalise '$f(x)=f(-x)$' to graphs with a line of symmetry different from the y-axis. Thus, the gap created by the emphasis in school was maintained.

[2] By DG microworlds I mean the sequence of DG Parallel to DG Cartesian.

Periodicity

Periodicity was also a property explored in the microworlds with Cartesian representations. Only two perceptions were explored in DG Parallel: a variational view by one pair and 'the periodicity between 'y follows x' and 'y does not follow x'' by two pairs. Conversely, by exploring both the contrast between 'absence of the shape' and 'visualisation of the shape when (x,y) moves' and the 'motions of x, y and (x,y)' in DG Cartesian, almost all the pairs developed a variational perception of periodic graphs. Thus, DG Cartesian comprised a bridge from their previous knowledge of periodicity to its perception in DG Parallel.

The transformations of graphs were explored by almost all the students to revise their previous hypotheses about period of sine function such as: two functions with the same period must have the same trace; the period is the interval between two roots; the period does not vary when calculated based on different 'special points'. A functional view of periodicity, which separates the variables, was also developed by one of the pairs while exploiting transformations of graphs.

2 Qualitatively different representations

Common perceptions among the pairs of students made clear the 'special status' attributed to some properties by 'the motions of x and y' in DG Parallel indicating an acquisition of a variational perception of function (Goldenberg et al, 1992) although this depended on the property in question. The students developed variational perceptions of turning points, constant function, monotonicity, derivative, second derivative and range and also applied them to a wider set of functions than they had previously. On the other hand, all the students presented difficulties in identifying other properties in DG Parallel such as symmetry and periodicity. In general these properties had previously been perceived by the students by reference to the shape of the graph.

Moreover, the work in DG Parallel, sometimes mediated by the work with DG Cartesian, helped the students to develop variational interpretations of some of these properties in Cartesian representations. For example:

- turning points started to be identified as 'point where y changes orientation';
- horizontal straight lines were justified by 'y is independent of x';
- monotonicity as direction of straight lines was interpreted by 'comparing orientations of the motions of x and y'. Two of the pairs of students integrated the

generalisation of monotonicity to quadratic functions from DG Parallel to Cartesian systems;

- slope of linear graphs was discriminated by 'comparing the ratio between the variations (or values) of x and y', linking it to inclination;
- on trying to interpret 'ratio between the variations of x and y' in parabolas, the curved and straight graphs were characterised and justified by constant and variable 'ratio ...';
- the perception of range in graphs changed from a polarised approach (positive and negative range) to an approach involving 'bounded or boundless range'.

The students' explorations of qualitatively different representations, embodied in DG and FP microworlds, showed that the variants and invariants by reference to which the properties were instantiated and identified were different in each microworld. The students' abstractions, using the term of Noss & Hoyles (1996) — their situated abstractions — were derived from the features of the microworld such as the students' articulations of derivative as 'ratio between variations of x and y'. The interactions with DG microworlds supported situated abstractions with variational aspects rooted within 'the motions of x and y' and the topological aspects of these motions. On the other hand, the interactions with FP mainly shaped students' perceptions by drawing their attention to *new*[3] aspects of their previous pictorial perceptions through the graphs and by instantiating these new aspects in the variants and invariants of the transformations. The students' abstractions were expressed through the tools of the microworlds but also shaped by the activities of the microworlds — descriptions and guessing activities. The students gave their 'new' perceptions non-mathematical terms which were frequently linked with the program and/or their partner's language: motionless for constant functions in strikers; progressive for increasing graphs; and roller-coaster for periodic graphs. Thus, as we should expect, the syntheses were not a direct process of 'translating' each point learned from one microworld into the other.

This difference in the students' development from property to property contributed to the discussion on the validity of the use of multiple representations in approaching the concept of function. The students easily discriminated some properties in DG Parallel but others such as symmetry were discriminated with difficulty. They also explored DG Cartesian to search for the functional meaning of the properties which they had not discriminated in DG Parallel. Nonetheless, these searches depended on previous pictorial perceptions of these properties derived from knowledge in graphs. This shows that a key to the use of multiple representations is allowing students to

[3] By new aspects I mean the aspects which the students did not previously know.

express and generalise their perceptions in their own way within a medium — to articulate situated abstractions (Noss & Hoyles, 1996) — meanwhile providing activities which lead them to synthesis.

3 Synthesis

Connections depends on the property and the representation

Ferrini-Mundy & Graham (1994) showed that students' ability to connect ideas between algebraic and Cartesian representation varies from property to property. My study took their results further by showing that the connections between the microworlds forged by my students depended on the property. This suggests that for any multiple representational environment the connections vary with the properties. Nonetheless, this statement needs further investigation.

Moschkovich (1992) reported that her students usually made connections in multiple representations by matching properties identified by the 'same status' such as 'special points' in graphs to 'coefficients' in equations. My results differ from that because few connections of this sort were forged. Moreover, these few occasions were first attempts, and in most cases later revised. The exploratory nature of the microworlds encouraged the students to realise that the same property could be recognised differently in different representations, which is one point Moschkovich (1993) assumes to be essential for students in making connections. Two types of 'special status' were identified: the same object and the same adjective. Both types are ways of using invariants to connect properties; the invariants are the objects in the first case and the adjectives in the second.

DG Cartesian as a two-way bridge

Interactions in DG Cartesian helped the students to make connections with perceptions derived from DG Parallel as well as with their previous knowledge. Nevertheless, they did not always consider these perceptions to be two different perceptions of the same property in DG Cartesian. When the connections were made, two kinds of behaviour were observed:

- perceptions constructed in DG Parallel were used to understand properties in the Cartesian system mainly for turning point and properties related with variation;
- previous pictorial perceptions recognised by the shape traced by (x,y) were brought into the discussion and a functional correspondence was sought using the relation between x and y.

In both cases, DG Cartesian was used by the students as a bridge between DG Parallel and the Cartesian system. Goldenberg et al (1992) hypothesised that the exploration of the sequence Parallel to Cartesian version of DynaGraph would work as a bridge to bring variational perceptions developed within the Parallel version of DynaGraph to the Cartesian system. The results of the present study go further by showing that DG Cartesian was used as a 'two-way' bridge.

The activity of guessing how a transformation in graphs affects the strikers, suggested by Kaput (1992) to motivate connections, was also useful in the search in DG Parallel for perceptions of properties which were not previously observed in DG Parallel or in Cartesian System. This indicates a different activity for building a variational analysis of the Cartesian representation which can be added to the suggestion of Goldenberg et al (1992).

Activities which led students to connect

In DG Cartesian the students made connections mainly while:

(a) recognising the family of functions to which the strikers belong;

(b) bringing terms studied in each family of functions to make sense in DG Cartesian.

(a) is a spontaneous version of one of the activities suggested by Kaput (1992) to lead students into connections — matching 'objects' from different representations. (b) is a different one involving the use of mathematical terms which is an important aspect of their school mathematics. Both schemes together with the fact that in DG Parallel only the pairs who previously worked in FP made connections show that the same functions in similar representations helped the students to complete the bridge between the Cartesian representation and DG Parallel.

In FP, the synthesis took place while the students were:

(a) using terms which generate discussions about their meaning;

(b) analysing the variants and invariants while transforming graphs;

(c) distinguishing two or more functions;

(d) observing algebraic and Cartesian representations of a function while transforming graphs;

(e) making sense of results obtained from transformations of graphs which were counter-examples to their own beliefs.

Once more the motivation of promoting links by discussing mathematical terms appears in FP. Motivation (b) is argued by Borba & Confrey (1992) to be one way by which students develop and strengthen their understanding in FP. Moreover, motivations (b), (d) and (e) are directly linked to interaction with transformations of graphs and show the importance of this feature not only for revising and building perceptions but also for building connections. Motivations (b) and (d) highlight the importance of the dynamic process of transformations in building the connections, instead of having only the starting and ending graphs of the transformations as discussed by Borba (1993). In these cases, the synthesis took place during the transformations of a graph. In the case (e), a different moments emerges as prompting students into connections: when their own beliefs were contradicted by the transformations.

The two activities suggested by Kaput (1992) to promote connections, used in the final interview, were crucial for the students to articulate connections: first, matching the strikers and the graphs and second, guessing how a transformation in a graph affects the corresponding striker. Thus, the findings show that they composed efficient ways of promoting bridges which seems to be a preoccupation of those who work with 'new' representations. Together the connections built in these two activities and the mechanisms used by the students to build the connections all suggest forms of creating bridges (Gurtner, 1992) free of the constraints of the teacher's or researcher's perceptions.

4 Associations

Origins of the difficulties and associations

Most of the associations originating in the research environment reflect a legitimate way of recognising a property among a limited group of functions. Almost all these associations were revised during the interactions resulting in perceptions applicable to a wider set of functions. This shows once more the importance of analysing students' perceptions in an 'alternative concepts' approach (Moschkovich, 1992). The origins, usefulness and limitations of these perceptions and the moments at which students were encouraged to revise them are the main points of this analysis. Associations were shown to form a natural process in the construction of knowledge and had to be investigated in their positive and negative aspects.

Other associations clearly coincide with the emphasis the school gives on the topic of function. The results show that stressing one property in one family of functions or restricting the study of a property to one family of functions led the students into associations which were valid only for these functions. Moreover, these associations impeded the development of students' perceptions. These findings include the prominence given to linear functions reported by Sierpinska (1992) and analysed in respect to school mathematics by Markovits et al (1983) and Schwarz & Hershkowitz (1996). This emphasis led the students to stop generalising perceptions among different families and to stop investigating new characteristics for that family.

I identified some tendencies in the associations which students presented while dealing with functions: interpreting properties pointwisely particularly on special points; transforming a property into a rule which polarised knowledge; and using the same object or the same adjective as an invariant to connect different properties as being the same.

Interaction with DG microworlds and a pointwise view

In my study, associations were originated from a tendency to analyse graphs in a pointwise way for the properties of monotonicity, derivative, second derivative and periodicity. In the school curriculum the students begin with a pointwise view in graphs (Goldenberg, 1988, 1991 and Monk, 1992). However, the main point is that for monotonicity, derivative and second derivative the interaction with the microworlds led all the pairs of students to revise these pointwise associations while for periodicity only half of the associations were revised. These findings also show a contrast with the results obtained by Goldenberg et al (1992). Although this tendency was present, the further interactions with DG microworlds together with the description/guessing activities stimulated the evolution from a pointwise to a variational perception for the properties of variation even with the presence of scales. This shows the great importance of using DynaGraph in allowing the students to develop variational ways of analysing graphs in a representation close to the Cartesian system.

The properties of variation were revised in more cases than those of periodicity and linearity. As these last properties were emphasised at school by the use of special points, I suggest that a barrier derived from knowledge ‘well’ established at school as special points is harder to overcome than a preference for following a pointwise way of analysing function.

Polarisation of knowledge as the main obstacle in the microworlds

Artigue & Dagher (1993) reported that their students preferred knowledge to be polarised. In my research, this preference was even more marked, and furthermore, my students transformed properties into rules involving polarisation such as positive versus negative and adopted a polarised way of analysing the properties.

The fact that it was revised in only half of the cases in the final interview, when syntheses were motivated, shows that the interaction with all the microworlds did not help in overcoming the tendency to use polarised perceptions, as it did with the pointwise ones.

Counter-examples in revising the associations

Students revised associations mainly when analysing counter-examples to them. Some of these counter-examples were taken from the twelve selected functions and others were generated by the students using transformations of graphs in FP. The description/guessing nature of the activities had an important role in the process of analysing within a 'function'[4] perceptions built for another 'function'. This process prompted the students to analyse counter-examples. Here, the study mainly distinguishes the exploration of FP, where the students were able to generate examples from a given function, from that of DG microworlds, where this was not a possibility. FP allowed the students the flexibility to seek counter-examples, which is a use of computers suggested by Dubinsky & Tall (1991). In DG microworlds, in contrast, the students were limited to the examples given by the 'researcher'.

Interaction with FP helped the students to re-integrate knowledge

In 'didactical transposition' (Chevallard, 1985) of knowledge has been compartmentalised to be put in a linear sequence. This was pointed out by Dreyfus & Eisenberg (1990) as one of the causes of students' reluctance to visualise. My results showed that the interaction with transformations of graphs while investigating different properties led two of the pairs to integrate properties by perceiving relationships between them. However, more research is necessary to verify whether this integration affects the reluctance to visualise. The results only showed that the students who began by working with FP linked the strikers with graphs, instead of equations, indicating a shift to visual thinking.

Lenses which make the associations observable

According to Confrey (1992a) dynamic transformations of graphs provide researchers with access to the processes of visual reasoning about shape and location when students fit a prototype function into desired points. The present research carries this further by showing that the limitations and associations of the students' previous perceptions emerged more clearly during an investigation using transformations of graphs. This access was also apparent in DG Parallel, a 'new'

[4] Function here means strikers in DG microworlds and graphs in FP.

representation, which did not show the shape of a graph. Thus, for different reasons, FP and DG Parallel acted as lenses revealing different aspects of the students' previous knowledge.

5 Overcoming the limitations of associations and obstacles

Working within DG microworlds

The study showed that by articulating situated abstractions (Hoyles & Noss, 1993) within DG microworlds, the students developed perceptions robust enough to challenge previous perceptions derived from school knowledge. Perceptions were developed using students' own created language which enabled them to generate ideas independent of their previous mathematical knowledge and terminology. The study also showed that when freed from mathematical language and previous constraints, the students had the potential to generalise some perceptions to a wider set of functions within DG microworlds, for example: on generalising monotonicity as 'y follows x' to non-linear functions. Moreover, integration of knowledge (Schwarz & Dreyfus, 1993) was observed when the perceptions already generalised were connected with mathematical knowledge. The students generalised the property also in the Cartesian representation. In contrast, for the pairs of students who earlier connected these perceptions built within DG microworlds, such as 'y follows x' with the term 'increasing', this generalisation was obstructed by previous knowledge. The new perception was not robust enough to challenge the previous one.

The findings of the final interview also showed the validity of allowing students to articulate perceptions within a 'new' environment such as DG Parallel. In the final interview, the students:

(a) generalised perceptions previously restricted to one family of functions;

(b) connected corresponding perceptions which remained isolated in different microworlds.

So, this validity was shown even when direct links were not observed as in case (b). Case (a) also shows that these articulations of situated abstractions (Hoyles & Noss, 1993) can be helpful in leading the students to overcome limitations of previous perceptions. Thus, the environments presenting 'new' representations can best be used for activities which enable students to build their own bridges.

Concentrating on graphs

The obstacles reported by Artigue (1992) in students' use of equations did not appear so clearly in this study. The students were blocked by the presence of equations only twice. The fact that the changes in methodology were designed in order to switch the students' attention from equation to graphs in FP seemed to lead them to concentrate on graphical features.

Considering the variables

As in the work of Sierpinska (1992), the students started without specifying the variable they were considering and this generated many associations. However, two aspects of the research environment led them to change: the description/guessing activities led them to require such specifications of their partners; and the separation of x, y and (x,y) in DG microworlds helped the students to identify which variable they were talking about.

Goldenberg (1988) and Clement (1985) showed that students usually interpret graphs in a pictorial way only. The facts that DG Cartesian presented x, y and (x,y) separately and that it allowed visualisation of the shape of a graph encouraged the students to seek functional meanings of the properties which they previously perceived pictorially in graphs: line symmetry and periodicity. The functional perceptions of periodicity and line symmetry were obtained by the majority of students from a correctly mathematical viewpoint. Even in the case of developing a mathematically incorrect meaning, these searches represented qualitative changes in the students' interpretation of graphs — from pictorial to functional.

6 Interacting with transformations of graphs

Goldenberg (1991), Kaput (1992), Confrey (1992a), Confrey et al (1991b) and Borba (1993) claim that the transformations of graphs should be done within the Cartesian representation in a dynamic way. The present study revealed patterns in students' development of their perceptions of different properties by interacting with the transformations. The students used the transformations to:

(a) generate hypotheses and check them by generating examples and counter-examples which allowed them to recognise and revise associations and, thus, following their own path of learning;

(b) discover new aspects of a known property as well as discover new properties;

(c) overcome limitations imposed by compartmentalisation of knowledge presented in the school mathematics;

(d) link properties by the observation of variants and invariants, in particular between algebraic and Cartesian representations;

(e) recognise the limitations of their perceptions and generalise them among different functions;

(f) develop a comparative measure for properties they previously perceived pictorially.

Some of these uses were already reported by Borba (1993) and Confrey et al (1991b). Together (a) and (b) are similar to potential computer use by mathematicians: generating data which suggests theorems (Dubinsky & Tall,1991), where hypotheses replace 'theorems'.

On the other hand, the link between invariant properties while transforming graphs also generated perceptions which were valid only for one family of functions. This shows that transformations can also generate negative aspects for students' perceptions.

The importance of the dynamic aspect of the transformation of graphs can also be observed in the fact that reflections were explored less than translations and stretches as was the case in the results of Borba (1993).

Two facts show the importance of acting on Cartesian representation by transforming graphs to lead the students to focus on qualitative properties of functions:

- two of the students used the mechanism of the transformations while later sketching graphs with paper-and-pencil abandoning their previous way of plotting graphs;
- the students who began by working with FP tried to connect the behaviour of the strikers with graphs, instead of with equations.

These findings show that on changing the status of the Cartesian representations into action representation (Kaput, 1992), the students started to attribute to the Cartesian representation the same importance as the algebraic one. Finally, they show how the mechanism of transforming graphs can scaffold a way of sketching graphs (Hoyles & Noss, 1987, 1993 and Noss & Hoyles, 1996).

X — Conclusions

1 Summary of research

My aim to investigate students' perceptions of function as they interact with different dynamic representations of function available through computer environments led me to design empirical research, to be undertaken in Brazil, comprising case studies with four pairs of students.

Perceptions of function were investigated through the evolution in students' perceptions not of the concept of function itself but of a variety of properties of function such as turning points, constant functions, monotonicity, derivative, second derivative, range, symmetry and periodicity. This variety covered different ways of analysing functions as reported in the literature: pointwise, variational, global and pictorial. Thus, this research focused on the different ways in which the students perceived each of the properties in the different dynamic representations.

The potential of dynamic computer environments was analysed in order to select two software programs which exploit the possibilities of computers to explore representations of functions by continuous movement: DynaGraph (Goldenberg et al, 1992) and Function Probe (Confrey et al, 1991a). DynaGraph allows students to vary-the-variable of a function and observe the variation of its image. Function Probe allows continuous and direct transformations of graphs, which change the status of the Cartesian system into an action representation. Thus, the research was designed specifically to investigate how the dynamic tools of DynaGraph and Function Probe might structure students' perceptions of the selected function properties. Considering the importance of the activities in any interaction with a medium, both programs were used in the creation of microworlds consisting of the software tools and a set of activities.

The design of the microworlds involved: the selection of twelve functions which emphasised the properties and allowed exploitation of the dynamic potential of the software programs; elaboration of activities of description/guessing and classification of the functions which led the students to explore the function properties while interacting with the microworlds, to develop a language and to discuss between themselves; adaptations of DynaGraph, DG Parallel and DG Cartesian, to enable exploration of the selected functions without the students having access to the corresponding equations.

In order to investigate the use of these microworlds against a background of the Brazilian curricula, this study was undertaken with Brazilian students who had already studied functions at school. Thus, a pre-test and an analysis of the school approach to function served as starting points. Both focused on the chosen properties and revealed students' previous perceptions and some over-generalisations and barriers. This allowed me to discuss similarities between epistemological obstacles revealed by the research activities and the school approach to function.

By working with multiple representations of function, the study investigated how the students came to discriminate and generalise each of the function properties within each of the microworlds. It also investigated the syntheses made between perceptions derived from activities in different microworlds and those constructed in school. A final interview was undertaken to investigate links students made during the activities as well as to motivate synthesis where possible.

A longitudinal analysis was undertaken tracing the evolution of students' perceptions of the function properties while interacting with the microworlds, giving consideration to the origins of these perceptions, any limitations and the set of functions to which these perceptions could be applied from a mathematical viewpoint. This analysis attempted to identify the main aspects of each of the microworlds which appeared to contribute to the students' progress. To do this, a purpose-built methodology was devised which culminated in the development of a visual presentation of a longitudinal analysis of this kind — the blob diagram.

After the longitudinal analysis, the findings for each of the pairs of students were summarised in a cross-sectional analysis focusing on: any links made by the students while interacting with the microworlds; different ways in which they appeared to provoke connections; any patterns in the students' development of their perceptions.

2 Contributions to mathematics education

I will discuss the research under two headings:

- the methodology and its design;
- the findings.

2.1 The methodology

The activities designed for this study required of the students different perspectives on a variety of function properties. The criteria used in the selection of the sample of functions were also important in the design of the activities. The analysis of each property of different functions played an essential role in leading the students to generalise and revise perceptions of each of the properties. A process of pilot studies and analysis of their findings searching for possible restrictions on students' perceptions showed the importance of a careful selection of examples and counter-examples of the concept. The fact that the students most often revised associations when using counter-examples also showed the importance of this selection in designing activities for a longitudinal study.

Meanwhile, the careful arrangement of the description/guessing and classification tasks led the students into peer interaction not only by direct discussion, but also by investigating the partner's perceptions. These investigations also led the students to:

- investigate the applicability of perceptions developed in one function within different functions leading to generalisations and revisions;
- negotiate common perceptions;
- describe precisely their perceptions when requested by the partners, all of which led to an identification of variables in the functions.

Thus, these activities comprised a methodology for revising and generalising perceptions involving peer interaction and thoughtful exploration of computer environments.

In the development of the activities DynaGraph was adapted to allow exploration of the twelve selected functions without access to their corresponding equations. Thus, it can be used in a first introduction to the concept of function when students do not have any knowledge about functions and also in a later exploration linked with known representations to lead students into variational and topological perceptions of functions.

Finally, the UDGS model of analysing students' understanding was adapted to consider three of its phases (discrimination, generalisation and synthesis) in terms of perceptions of function properties within and between microworlds (representations) and qualitatively different functions with respect to properties such as different families of functions. The analysis focused on the mathematical aspects of students' perceptions. Also, the analysis of limitation, origins, and applicability of each perception when placed in the sample of functions throws light on each property under consideration.

One of the problems while analysing the longitudinal study was the visualisation of students' progress in the whole process throughout different microworlds. The methodology developed culminated in a visual presentation of the evolution of perceptions of a concept — the blob diagram. This diagram emerged as a tool for analysis as well as for the presentation of the analysis, allowing visualisation of:

- isolated perceptions;
- continuity in the process of constructing an idea;
- revision and generalisation of perceptions;
- connections between perceptions built in different microworlds;
- use of DG Cartesian as a bridge between perceptions in other microworlds;
- difficulty of perceiving a property in a microworld;
- dominant perceptions;
- the path which each of the perceptions traced through the sequences of microworlds;
- perceptions blocked by others.

The diagram also presents a historical analysis which includes perceptions from the pre-test to the final interview for which a post-test could be substituted. Moreover, its design is easily adaptable to the number of microworlds or settings of further studies. The use of this diagram allowed me to extract the main points of students' perceptions from the detailed analysis of them throughout the empirical study.

2.2 The findings

The Brazilian curricula

The findings of this research point to some implications for the way functions are introduced in the Brazilian mathematics curricula. Limitations were found in the students' perceptions of the function properties and also barriers identified which seem to be derived from the school approach. The effect of the students' school knowledge on their perceptions of the properties was observed mainly in DG Cartesian and FP while analysing associations made and the obstacles faced. Two tendencies were revealed: the students started by using pointwise perceptions; and they polarised knowledge in their analysis of the function properties. The emphasis the school gives to polarising knowledge led to barriers against generalisation and revision of perceptions: the study of inequalities apparently led to an approach to function which posits all knowledge in terms of positive and negative. This could in fact be interesting for exploring inequalities but it led the students into difficulties while analysing the function properties.

Another limitation of the curriculum is the absence of any work leading students to compare different functions within and between families which seemed to lead to some revisions of associations. Emphasising a property for only one family or a particular set of functions, such as monotonicity for linear functions and range for bounded functions, led students to develop associations. These were valid as a way to discriminate the property among these functions but came to replace the meaning of the property. The associations were not derived exclusively experiences in school which emphasised linear functions but also from those which stressed other families of functions.

Mathematical terms were in general used by the students only for the families of functions in which they were emphasised at school. The use of these terms was in general linked with rules of recognition, thus making them more difficult to revise. These rules appeared implicitly in the textbook, used as basic material, and were given as ways for the students to recognise the property. In fact, they proved to be substitutes for the meaning of the property. Moreover, sometimes these rules were used by the students to generalise properties among different families of functions. This led the students into over-generalisations such as 'an increasing parabola is a parabola with positive curvature'.

Variational perceptions developed in DG microworlds

This research showed that the exploration of the way DynaGraph represented functions led the students to develop a variational perception of some of the function properties as well as to focus on topological aspects of other properties. In fact, the possibility of manipulating x and observing the motion of y enabled the variational aspects of a function to become properties with special status. These variational aspects together with topological ones enabled the students to generalise properties, such as monotonicity, previously restricted to only one family of functions. The explorations also led the students to become aware of qualitative aspects distinguishing different families of functions such as the constant and variable derivative for linear and non-linear functions. Nevertheless, the research also showed how hard it was for the students to discriminate in the parallel version properties such as symmetry.

The interactions with DG microworlds led the students to change their preference for pointwise perceptions but not their preference for polarised knowledge. Nonetheless, the topological way of analysing properties helped the students (in cases such as bounded range) to abandon polarisation in their analyses.

Interactions with DG microworlds led to specifications of variables

The students started to specify the variables which they were talking about while analysing functions in DG microworlds. Two aspects of this research brought about this change: the nature of the describing/guessing activity and the fact that DG microworlds presented separately the objects x, y and (x,y).

DG Cartesian serving as a two-ways bridge between variational and pictorial perceptions

The use of DG Cartesian provided an interesting halfway representation between Cartesian system and DG Parallel because it enabled the students to perceive the function properties variationally and also through graphs.

The study also showed that DG Cartesian was explored by the students as a two-way bridge between variational perceptions built in DG Parallel and pictorial perceptions from the Cartesian representation. Not only were the properties with special status in their variational perceptions in DG Parallel synthesised with the Cartesian system, but also the properties which the students knew pictorially in the Cartesian system were connected back into DG Parallel. The explorations of DG Cartesian led the students to search for functional meanings of pictorial perceptions because of: the presentation of the variables x, y and (x,y) as separated objects; the description/guessing nature of the activities; and the contrast between the shape of a graph visualised by the motion of (x,y) and the absence of its trace on the screen.

Articulating situated abstractions in DG Parallel

Given that few attempts were made to build connections to 'old' knowledge during interactions with DG Parallel, this microworld could be explored as a 'new' representation where students appeared more free of previous perceptions. This allowed them to revise and generalise perceptions within this microworld. In the case when later connections with previous knowledge were made, the developed perceptions proved to be robust enough to allow students to contrast them with those derived from school knowledge. Thus, a key to the use of qualitatively different multiple representations is synthesis but also articulation of situated abstractions.

Transformations of graphs as means to explore properties

In FP the students' interpretation of the properties could not really be categorised in relation to each command explored. The research found patterns of similarities in the students' perceptions and the commands explored only for derivative. The point of exploring the transformations of graphs in FP was to give the students tools to explore, not to shape conceptions — in contrast to DG microworlds. The fact that the students were discussing while transforming graphs more often determined the changes in their perceptions, than the command per se. The commands were used to support the investigations of their hypotheses. Thus, what was revealed in this research were patterns emerging from the ways the students used the commands to modify their own perceptions. The students used the transformations as tools to generate and check their own hypotheses by generating examples and counter-examples, thus recognising and revising differences in perceptions previously associated and discovering new aspects of a property. They also realised the limitations of their own perceptions and generalised them among different functions. The use of the transformations also enabled students to develop 'comparative measures' for properties they previously perceived pictorially and to realise relationships between different properties which had previously been compartmentalised.

The research showed on the other hand that by observing variants and invariants of transformations, the students were also led into perceptions valid only for limited families of functions.

Interactions with FP changing preference for graphs

Transforming graphs enabled the students to extend their skills in building graphs and to modify their preference for visual representations. It showed that the students who began by working with FP linked the strikers with graphs while the others linked them with equations. This also demonstrated a change in the obstacles the students faced by considering equations as being 'the essence of a function'.

The change in the status of the Cartesian representation from feedback to action representation altered the students' preference for visual thinking. Nonetheless, what is really interesting is that the students who began by working with FP, after leaving this microworld, used mechanisms similar to the transformations to sketch graphs of the strikers. They concentrated on qualitative features of each family of functions to carry out translations, stretches and reflections on a prototype graph.

The microworlds as lenses to reveal associations

This research also showed how the microworlds were tools for clarifying associations and important aspects of a property which at times stood for the meaning of the property. Thus, they were used as 'lenses' for searching for important aspects emphasised in each of the properties. For example, without the shape a 'linear' function becomes a 'function which passes through (0,0)' in DG Parallel. In FP, the emphasised aspects were revealed in the hypotheses the students were generating and checking while transforming graphs.

Patterns in ways of synthesising

It is possible to come up with some patterns in the ways which led the students to synthesise. In the case of DG Cartesian, the students made the connections by matching the strikers with the family of functions and bringing terms explored at school in these families to the discussion. This last was also exhibited in FP. The fact of working with the same sample of functions in different microworlds also encouraged the students to make connections.

In the case of FP, the research showed that the students were more open to making connections in response to: the analysis of variants and invariants and the observations of algebraic and Cartesian representation while transforming graphs, which then showed the great importance of the dynamic transformations of graphs for the students in building the connections; the attempts to make sense of results obtained from transformations which were counter-examples of their own assumptions, which then demonstrated that students were stimulated into making connections when their expectations were contradicted; the comparison of two or more functions, which then highlighted the importance of the activities of describing, guessing and classifying functions in leading them to connect perceptions.

The two activities of the final interview provoked the students to make their own connections by linking perceptions which before had been isolated within different microworlds, generalising perceptions previously restricted to one family of functions and revising naive links. The activity of predicting a striker corresponding to a transformed graph also led the students into a new search for perceptions in DG Parallel which they brought to the research environment by Cartesian representation.

The nature of the activities leading to the results

The results of this research depended not only on the computer features but also on the students' interactions during the activities. One illustration of this can be given by the fact that the development of ways to measure, such as ‘ratio between the variations of x and y’ or ‘distance between two symmetrical points’, is directly linked with the description and guessing nature of the activities. The students had to be precise in comparing two or more functions in order to allow their partners to guess the function described. In the case of DG microworlds, I also believe that the presence of scales also encouraged them into the above-mentioned measurement systems.

3 Limitations observed in the research

While observing the sessions and analysing the data of the study, limitations were observed in relation to the software programs, the activities designed and the research itself.

3.1 The microworlds

There are some limitations in the DG software programs. They are still running slowly, mainly while comparing two or more strikers in such a way that the order of the execution of each change can be observed. On some occasions this led the students to believe the one which moved first to be the quickest. Also, the programs still limit the user to work with a fixed scale and 'step of variation of x' and in addition the user (teacher) has to access the program to alter the selected functions. Thus, technical improvement is needed.

The activities presented some limitations imposed by the number of functions explored and therefore the associations generated. As the functions were chosen to highlight the properties as well as to provide counter-examples of associations observed in the pilot study, the use of only two functions was not always enough to lead the students to explore periodicity. Moreover, the absence of an oscillatory and aperiodic function caused the students to fail to distinguish oscillation from periodicity. For the same reasons, I would like to make clear that as it is a microworld with previously fixed functions, many associations which appeared during the development of the activities had no counter-examples for the students to contrast with. In the case of FP, these limitations were not so difficult to overcome as in DG microworlds because the students were able to generate new functions from the given ones. Nonetheless, when an association requested as counter-example a function from a different family, these limits re-appeared.

Also, on being requested to describe the twelve functions while describing and guessing the strikers, in the case of DG microworlds the students rarely returned to a striker already described to compare a new perception. These returns were more usually observed in FP as the students were describing graphs while exploring different transformations. This process could lead them to make more revisions. Sometimes, a counter-example of a perception could be provided by a function already described.

The Graph window of FP needed much work to make it accessible before it could be used by students unfamiliar with computers. The students always had difficulty in remembering how to operate each of the commands. Moreover, when operating with a mouse, they usually complained about the difficulty of placing graphs in the desired position. This problem arose when the students were asked to transform one function into another. They usually did this only into an approximation of the other. Operating a mouse with precision is not always easy. Finally, as manipulated by the students the software sometimes 'quit' unexpectedly which led me into difficulties in transcribing the students' work without the historical file of their explorations.

3.2 My research

Clearly a case study is limited and caution must be exercised before transferring its results to other contexts. In addition, it is clear that the study was not able to categorise all the perceptions and associations that might arise. Nonetheless, case studies provide a rich source of data into how students develop perceptions as well as provide a 'lens' for observing students' reasoning.

The methodology used in this study proved to be inadequate to address one of the research questions. I tried to find patterns of similarities between perceptions of properties and the transformations explored. On the one hand, the number of pairs investigated gave me a large list of different perceptions generated while interacting with the different transformations in the different graphs; on the other hand, these perceptions were too diverse for similarities between them and commands to be identified. In fact, this pattern of similarities was observed only for derivative. A study should be done with a larger group of students in order to address this question.

Finally, as the research was conducted outside the classroom, it was inevitable an artificial environment for the students. There, time was allowed for engagement, motivation and discussion and not constrained by the school time-table. Thus, any use of the microworlds in the classroom would necessitate adaptations which would probably generate differences in the results.

4 Implications for practice in Brazil

Despite recognising its limitations, the study provides a tool to use in case studies and a 'lens' to access probable obstacles to students' learning of functions.

The research also suggests ways of using the dynamic tools of these microworlds in Brazilian curricula. The activities can easily be adapted for use in the classroom to lead students into variational perceptions of functions specially while analysing graphs; into a more thoughtful exploration of their perceptions of the properties in graphs and into revision of perceptions. Such adaptation could constitute a concluding 'chapter' for the topic of function, for example, which encourage students to compare functions within and between families. Nonetheless, technical improvements in DG software programs are firstly necessary. Then, together with the mathematics teacher changes in the mathematics curriculum have to be proposed in the light of the problems revealed in the research. After that, the activities must be adapted to be used in the classroom. All these adaptation has to be analysed considering the possible effects of its use in the curriculum. Finally, as a research activity, a methodology of observations of students working in the microworlds in the classroom has to be examined. The results of such a study could be compared with those obtained in the present research and so reveal the differences in using the environment in the classroom.

The findings also suggest three different uses of the microworlds in isolation. First, the sequence DG Parallel to DG Cartesian can be adapted for a general introduction to the concept of function culminating in the introduction of the Cartesian representation. Nonetheless, a careful preliminary analysis of the consequence of removing a procedural approach should be undertaken, as well as the creation of opportunities for students to discriminate properties such as symmetry which are difficult to discriminate in DG Parallel. Second, DG Cartesian can be adapted to encourage students to search for functional meanings of properties pictorially perceived in Cartesian representation. Thus, this adaptation could constitute a revision of the topic of function. Third, FP microworld can be adapted to lead students into deeper exploration of graphs following their own paths and generating their own hypotheses. This work can be done within and between families of functions. This could change students' familiarisation with Cartesian systems and preference for visual thinking.

5 Future research

One result of the final interview, in which DynaGraph was used, led me to consider how the results might be affected if a striker were allowed to plot dots at each of the points as it jumps. One of the pairs easily differentiated constant and variable rate of change of linear and quadratic functions by these dots. Probably, the properties of symmetry and periodicity would be more easily discriminated. Also, the constant second derivative of parabolas would be distinguished from the variable one of sines. However, I believe that the emphasis on 'the motions of x and y' might be replaced by a pictorial view of the distribution of the dots. Moreover, by having the dots on the screen, the other side of the two-way bridge would not work. The students would not be interested in discussing their pictorial view in a functional way. Thus, some questions are raised: what would be the differences in the findings if DG microworlds left points on the screen? Or if the students were able to change scales, variation of x, and so on?

The imbalance between facilitating the discrimination of some properties and making it harder to discriminate other properties led me to consider the use the microworlds for students who had never studied functions and to ask how they might come to discriminate and generalise function properties when introduced to the topic by the sequence DG Parallel to DG Cartesian culminating with the Cartesian representation. This research led me to conjecture that they would exhibit variational understandings of the Cartesian representation. However, would they also develop pictorial perceptions? What about the synthesis between these two ways of perceiving properties in graphs? It seems to be clear that one side of the two-way bridge would not work. As the students would not have previous views of some properties, they could not search for functional meanings. Thus, what would happen with the properties which proved to be difficult to discriminate in DG Parallel, such as symmetry?

By answering these questions, I would be able to compare the results and evaluate whether the microworlds should be used as an introductory chapter or a concluding chapter for the study of function in the Brazilian curricula. This comparison would also allow me to go deeper into the obstacles faced for instance by 'new' students who would not have any previous knowledge in functions.

Going a bit further, as the results suggest, no sign of understanding the idea of limit was observed. For example, as in the Brazilian secondary curricula, derivative was perceived only as 'rate of change'. Thus, I wonder whether further these microworlds could be used to help students to understand functions in undergraduate courses of calculus. Would they change their perceptions of function properties? Would students' perceptions of the properties of variation get close to the idea of limit?

Finally, the research revealed that a microworld was used as a 'lens' for accessing different aspects of students' perceptions of function, it could help teachers evaluate the obstacles faced while studying functions and may be provoke them to evaluate their approach to functions.

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Appendices

Appendix I - Material of the Empirical Study

1 Questionnaire undertaken with the students

Questionnaire

1 - Name: ______________________

2 - Address: ______________________

____

____

3 - Telephone: __________

4 - Age: ______

____

5 - List the school courses that you like.

____

____

____

6 - Do you enjoy Mathematics? __________

7 - Do you have difficulties in studying Maths?

____

____

____

8 -Which topics of mathematics do you prefer?

____

____

____

9 - How long do you study maths in a week?

____

____

____

10 - Describe your mathematics classes. What sort of activities do you do there?

____

____

____

11 - Do you use group-work in maths classes?

____

12 -Have you ever worked in the same group as your partner?

____

____

13 -What role do you usually play in group-work?

____

____

14 - Do you use computers? If so how often do you use computers? Which sort of activities do you do with computers?

____

____

____

15 - Have you got a computer at home? __________

16 - Have you ever used computers at school? If so describe the type of activity you did with the computers?

____

____

____

17 - Do you play video-games?

____

____

____

2 The interview undertaken with the mathematics teacher

2.1 Interview about each pair of students

The questions below are a guide to structure the interview with the teacher referring to each pair of students:

1 - How do you evaluate A and B (the students) in Mathematics?
2 - Where would you rank them in class mathematics?
3 - Why do you think they are good/bad (an evidence of that)?
2 - What is their level of interest in classroom activities? And in maths exercises?
3 - Do they study hard for mathematics?
4 - How can you describe their participation in the classroom?
5 - Do they work together?

2.2 Interview about the approach taken on functions

1 - Can you describe your maths courses?
2 - Do you set home-work or just class-work?
3 - Do you use group-work, lectures, problem solving sessions or methods in class?
4 - Which curriculum material do you use in your classes?
5 - How do you structure your exams?
6 - How you did you work on functions with these students?

Then, I followed the interviews with topics of interest and investigation of further information he gave me.
- The role of the definitions in this topic.
- How he introduces function to the students.
- The emphasis given to each representation.
- The activities involved in building a graph and interpreting a graph.
- Use of contextual problems.
- Applications of functions in problem solving.
- How he teaches the students to plot a graph.
- If he makes the students compare functions.
- If he works with the functions separated into families.
- Which properties of function does he emphasise in each family of function, in particular those I'm using in my work.
- If he uses other representations.
- The curriculum material he uses to teach function to these students.

3 The pre-test

TEST

1 - Two cyclists — John (A) and Joseph (B) — are moving on a road towards the same point. Their respective distances from a gas station in relation to the time are given by the following equations:

$$D_A = t+6 \quad e \quad D_B = t^2-3$$

• A
| Gas station
• B

The time is denoted by (t) in seconds and distance is denoted by (D) in meters.

a) What are the positions of the cyclists at the starting moment, i.e. when t is zero? Who is ahead?

b) How long does it take for the cyclists to meet each other?

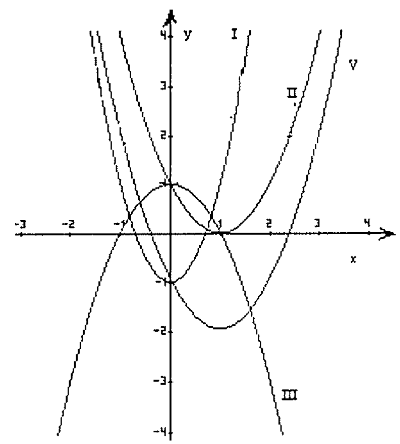
c) What are their respective speeds at the moment when they start?

d) Who is the faster cyclist when they meet each other?

e) What is the acceleration of each cyclist?

2 - What do you understand by function?

3 - Compare the curvature of the following parabolas. Give a growing sequence for their curvature.

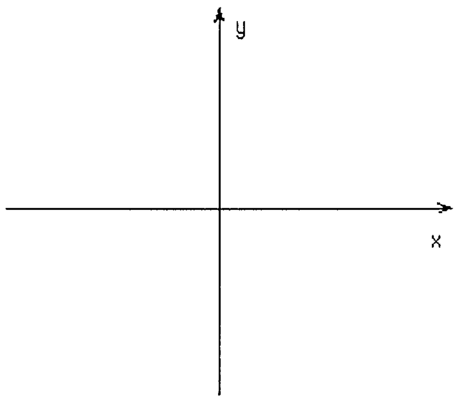


Increasing sequence of the curvature of the parabolas:

4 - What do you understand by velocity?

5 - What do you understand by acceleration?

6 - Try to sketch the graph of the functions given by the following equations:



a) $y=x^2$

b) $y=-2x$

c) $y=x-2$

d) $y=2$

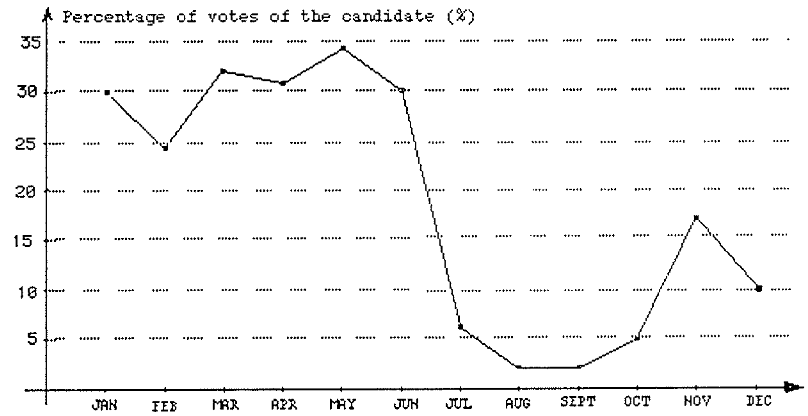
e) $y=4\sin(x)$

f) $y=x^2-4$

g) $y= x/2$

Identify the periodic functions by (P), the bounded functions by (L), and mark in the graph the turning points of the functions:

7 - The graph below represents the percentage of votes that a candidate for Mayor has during the year before the election in January.



According to the above graph answer the following questions:

a) During which period had the candidate less than 5% of the votes?

b) In which months did the candidate reach his/her maximum percentage of votes? What about her/his minimum?

c) In which months did the candidate obtain more than 30% of the votes?

e) What are the periods in which the percentage of votes of this candidate decreased?

f) What are the periods in which his/her percentage of votes had its biggest increase?

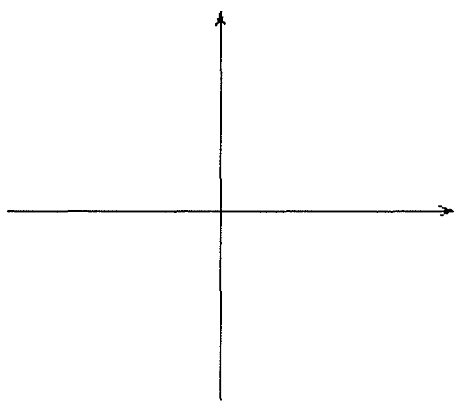
8 - Try to sketch a graph of the distance travelled by a car from a starting point in relation to the time spent in accordance with the following facts:

a) The car starts from a point. It moves accelerating during a period of 20 minutes, after which, it had reached a distance of 10 Km from the starting point.

b) Then, the car keeps moving at the same velocity up to a distance of 20 km. This takes 30 minutes.

c) After these 30 minutes the driver decides to stop for 10 minutes.

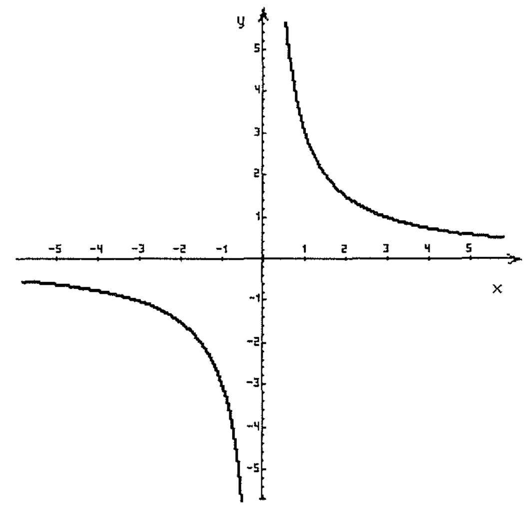
d) After relaxing, the driver returns to the starting point at a constant velocity. He takes 40 minutes to return.



9 - What do you understand by range of a function?

10 - Answer the following questions according to the function given by the graph and equation below:

$y=3/x$



a) What is the range of this function?

b) What is the domain where this function is increasing? What about the domain where it is decreasing?

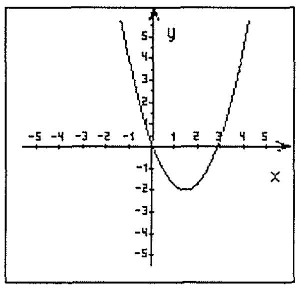
c) What happens to y when x is very close to zero in the positive side? What happens to y when x grows too much?

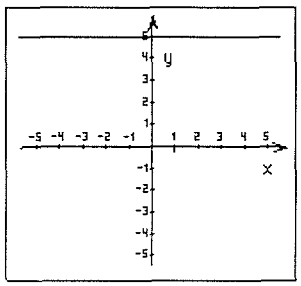
11 - What do you understand by increasing function?

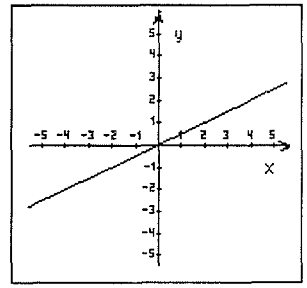
12 - What do you understand by periodic functions?

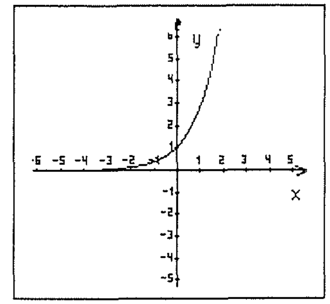
13 - What do you understand by turning point?

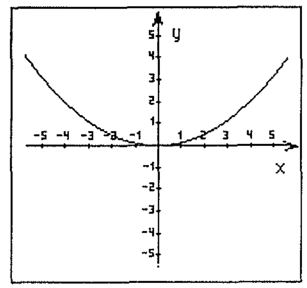
14 - Write down the range of the following functions. In the case of bounded function, write down its maximum and minimum as well.



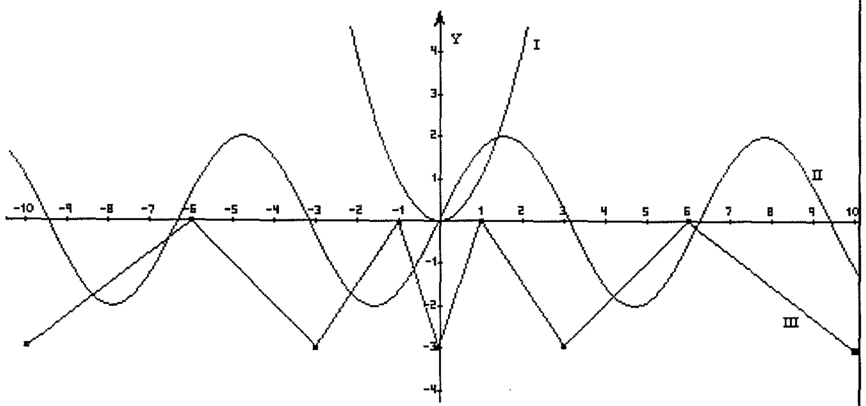








15 - Identify the periodic functions among the functions below. Identify also the symmetric functions with their line of symmetry:

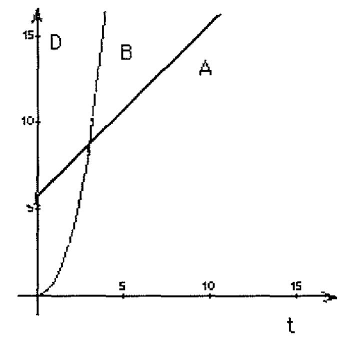


I -

II -

III -

16 - Two athletes Caio (A) and Andrew (B) are in a race. The following graph shows the respective distances of the athletes from the starting point in meters in relation to the time taken in seconds. Answer the following questions according to this graph:



a) Who starts ahead? What is the starting position of each athlete?

b) When do the athletes pass each other?

c) Who is running faster at the passing point? Why?

d) What is the starting velocity of each athlete?

e) What can you say about the speed of each athlete?

17 - What do you understand by constant function?

THE CAVE PIZZA

Menu

PIZZAS:

Personal	(12 cm diameter)	R$	3.00
Medium	(24 cm diameter)	R$	6.00
Large	(36 cm diameter)	R$	9.00
Super-Pizza	(48 cm diameter)	R$	12.00
Gigantic	(60 cm diameter)	R$	15.00

Toppings:

Four-cheeses, Mixed, Portuguese and Vegetarian.

Special opportunity:

You can pay for a pizza according to its area. In this new system, we charge R$ 0,01 per cm^2 of pizza.

A Journey through Function Probe software

As you can see in the menu, The Cave Pizza sells pizzas in several sizes. Thus, it tries to suit every preference. A personal pizza has 12 cm diameter, a medium pizza has 24 cm diameter, a large pizza has 36 cm diameter, a super-pizza has 48 cm diameter, and a gigantic pizza has 60 cm diameter. The prices of the pizzas in the menu are: R$ 3.00 the individual, R$ 6,00 a media, R$ 9,00 a large, R$ 12,00 a super-pizza and R$ 15,00 a gigantic.

The Cave Pizza also offers to the consumer an opportunity to pay for a pizza by area. It charges R$ 0.01 per cm^2 of pizza. You, as a consumer, obviously want to pay less for each pizza. Thus, we are going to explore Function Probe to compare the prices of the pizzas.

Part I - Table

Step 1. Starting the software

a) Open Function Probe by clicking twice (with the mouse) on the icon of Function Probe.

b) You will see three windows: Table '(tabela)', Calculator '(Calculadora)' and Graph '(gráfico)'. We are going to work first in Table.

c) Click once on Table to open it. Then, click on the zoom box (upper right corner) to enlarge the Table, as in the figure on the next page.

Step 2. Inserting a label in each column.

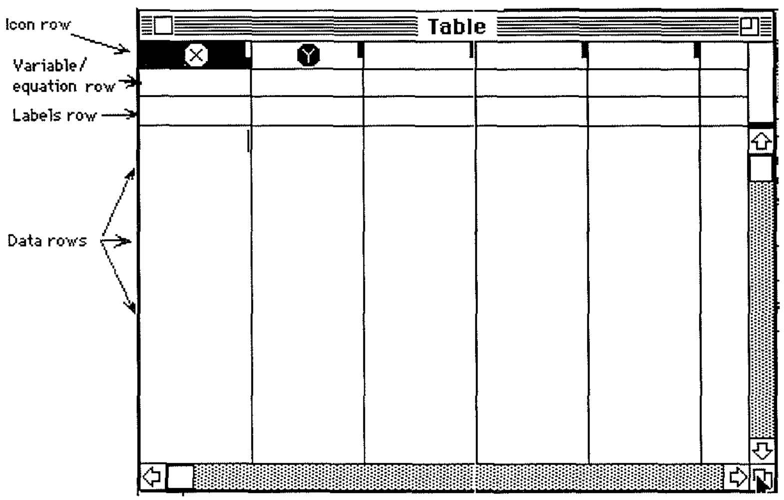
We are going to give a label to each column.

A) We are going to insert the titles of the pizzas in the first column, then we can label it 'Pizza'. Do it in the following way:

A.1) Place the cursor on the Label arrow and first column; and click once.

A.2) Type in: Pizza.

B) Repeat the instructions for the other columns, inserting the following labels: Diameter (cm), Radius (cm), Area (cm2), Price (R$); Pr. Area (R$) (price calculated according to the area of the pizza).



Step 3. Inserting data

A) First, Inserting in the data row and Pizza column the titles of the pizzas. For the next row press the **Return**-key.

B) In the Diameter column insert the diameter of each pizza.

C) In the Price column insert the corresponding price given by the Cave Pizza menu.

D) In order to fill the columns Radius, Area, and Pr. Area we are going to use formulas. As you know, radius is half of the diameter, a area is π times the radius squared, and for the Price calculated per area you have to find the formula. To fill the column with the formulas, we are going to:

D.1) Assign the variable D to the diameter column by writing the letter D on its Variable/equation row (Function Probe distinguishes D from d as different variables).
D.2) In the Radius column and Variable/equation row write down R=D/2. Thus, you are defining Radius as being half of the diameter of each pizza.
D.3) Repeat the instructions for Area and Pr.Area with the appropriate formulas.
Ob..: In order to obtain: π hold the **Option**-key and type **p**; R^2 type R^2.

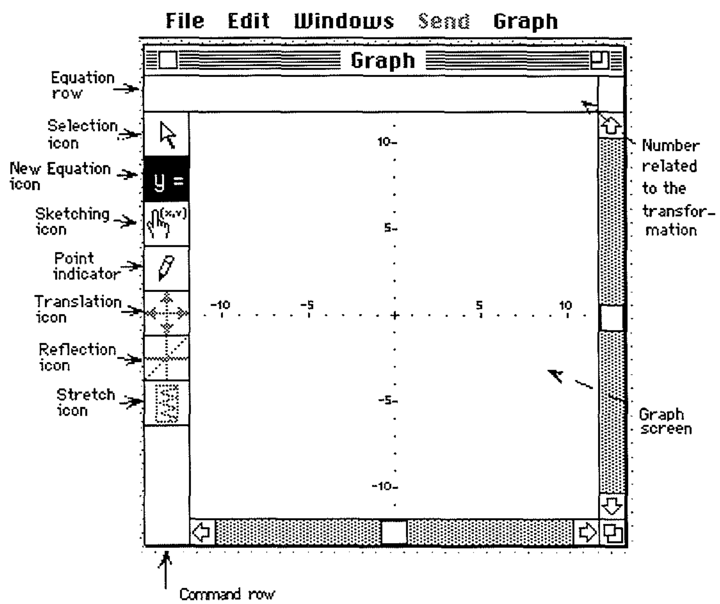
Question: Which way of paying gives the cheaper pizza?

Part II - Graph

Now, let us explore the Graph. We are going to graph the two systems of prices according to the diameter of each pizza.

Step 1. Selecting the graph window.

A) Place the cursor on the Window menu of Function Probe, click and hold the mouse-button, pull the mouse to select the Graph option, then release the button. Thus, the Graph window will be selected



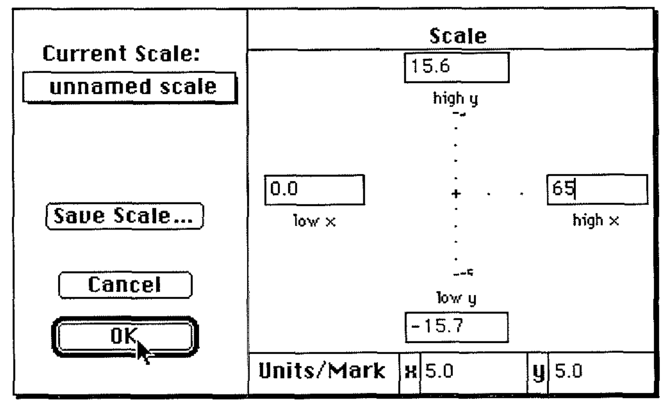
B) Click on the upper left corner to amplify the Graph screen.

Step 2. Plotting the points

A) Click on the point indicator, and move the cursor to the Graph screen. The ordered pair corresponding to the position of the indicator will appear in the Equation row . Place the cursor on the point (12,3) and click once. A dot will appear at this point.

B) As you can see, the graph screen is too small to plot the other values of diameters. Thus, before we continue, we are going to change the scale of the graph to enable us to see from 0 to 65 in the horizontal axis.

B.1) Click on the Graph option of the menu and choose **RESCALE** ('Mudar a escala').



B.2) Click on the square **low x** ('o menor x'), erase what is written and write 0, then on **high x** ('o maior x') write 65. Finally, click on OK. The Graph screen will appear from 0 to 65.

C) Now, plot the points of price of menu per diameter: (24,6); (36;9); (48,12); (60;15).

Question: Can you find the formula of a function whose graph passes through all these dots?

D) Click on the New equation icon; type in the formula and press **return**-key. If you have not found the correct one, try to adjust it using the transformations icons: Stretch, translation and reflection.

E) Click on the icon of the transformation you want and do the following:

Translation:

- Select from what appears in the screen the translation you want and click OK.

- Place the cursor on the graph. When the 'hand' closes, click (holding) and pull the graph to the sides or up and down.

- Release the click when the graph is in the desired position.

Reflection:

- Select from what appears in the screen the type of reflection you want to use and click OK.

- Place the cursor on the flashing line. When the 'hand' closes, click once.

Stretch:

- Select from what appears in the screen the type of stretch you wish to use, and click OK.

- Bring the cursor to the flashing line in the graph screen. When the 'hand' becomes an anchor, click once. Now, take the cursor to the graph, when the 'hand' closes, click (holding) and move the cursor to the sides, or up and down. Release the button when the graph is in the position you want.

Part III - Linking Table to Graph.

Let us trace the graph of the Price according to the area (in relation to the diameter) of the pizzas. This can be done by sending the points from the table to the graph.

Step 1. Adjusting the scale of y-axis.

As you can see in the Table, the price per area ranges from around zero to around 30, then select the **Graph - Rescale** and change the scale of y to 'from 0 to 30'. Try to do it by following the instructions for changing the scale of x.

Step 2. Defining x and y.

A) Change in **Window - Table** to select the Table window.

B) As the x-axis represents the diameter, let us define the diameter column with the icon . in the same way, The Pr.Area column will be defined with the icon .

B.1) Click on icon (holding) and pull it to the diameter column.

B.2) Click on icon (holding) and pull it to the Pr.Area column.

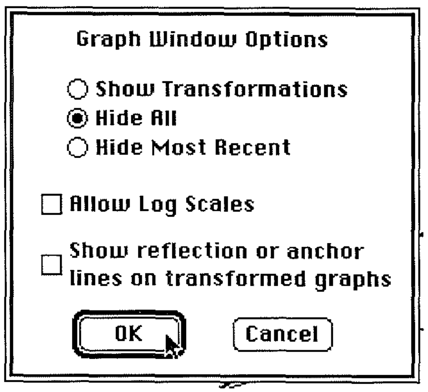
Step 3. Sending the points to the graph

A) Choose the option **Send - To Graph.** Then, change to the Graph window to observe the plotted dots.

B) Click on the New equation icon and write the equation $P=D^2$ and press **return**-key.

C) Try to transform its graph to fit the dots sent from the table, using the commands of Part II: translation, reflection and stretch.

D) Now choose **Graph - Graph options.** Change the option **Hide all** to **Show transformations** and click OK. In the Equation row of the Graph window the formula corresponding to the graph you obtained will appear, which should correspond to the price according to the area (per diameter) of pizza.



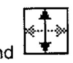
Question: The Cave Pizza wish to make pizzas in the size the consumer wants. For this, it needs two equations to calculate the two different prices. Try to find for which diameters of pizza the price of the menu is cheaper.

5 Worksheets and cards of FP microworld

5.1 Worksheets of the starting activity with FP

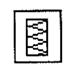
Worksheet 1

Notepad

1 - In the graph window of FP: use the command  to transform the function of y=abs(x)

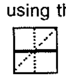
Function you obtained

Without using the equation, describe the function you obtained

2 - Use  to transform the graph of y=abs(x)

Function you obtained

Without using the equation, describe the function you obtained

3 - Use  to transform the graph of y=abs(x)

Function you obtained

Without using the equation, describe the function you obtained

Worksheet 2

Description

Starting function

y=abs(x)

Describe the function that you obtained

•
•
•
•
•

Compare the starting function with the function you obtained

SIMILARITIES

•
•
•
•
•

DIFFERENCES

•
•
•
•
•
•

My partner, can you find out the function I obtained?

5.2 Worksheets of the other sessions

Worksheet 3

Description

Starting function

Function tried

Did you obtain the function you tried? Yes ☐ No ☐

If so, compare both functions

Similarities:

•
•
•
•

Differences:

•
•
•
•

If not, what are the differences between the two functions?

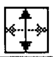
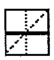
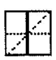
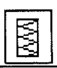
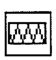
•
•
•
•

Worksheet 4

Worksheet of Effects of commands

Function

Which properties of the functions can you change with each command?

Commands	Change	Does not change
		
		
		
		
		
		

Describe the function above:

-
-
-
-
-

Worksheet 5

Description of groups

Group

Similarities

-
-
-
-
-
-
-
-

Differences

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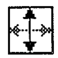
-
-
-
-
-
-
-

Worksheet 6

Worksheet of the effects of the command

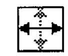


Which characteristics can you can change in each group of functions with the above command?

Function	Change	Cannot change

Worksheet 7

Worksheet of the effects of the command

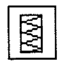


Which characteristics can you can change in each group of functions with the above command?

Function	Change	Cannot change

Worksheet 8

Worksheet of the effects of the command



Which characteristics can you can change in each group of functions with the above command?

Function	Change	Cannot change

Worksheet 9

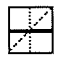
Worksheet of the effects of the command

Which characteristics can you can change in each group of functions with the above command?

Function	Change	Cannot change

Worksheet 10

Worksheet of the effects of the command



Which characteristics can you can change in each group of functions with the above command?

Function	Change	Cannot change

Worksheet 11

Worksheet of the effects of the command

Which characteristics can you can change in each group of functions with the above command?

Function	Change	Cannot change

5.3 Cards with the graphs

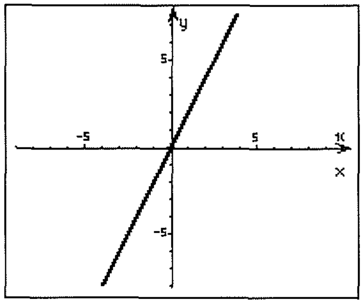


Grafico A

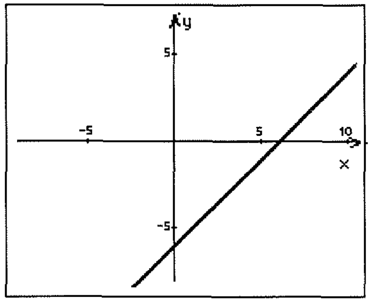


Grafico E

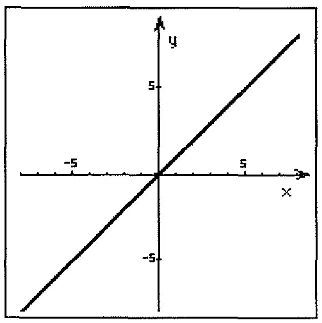


Grafico L

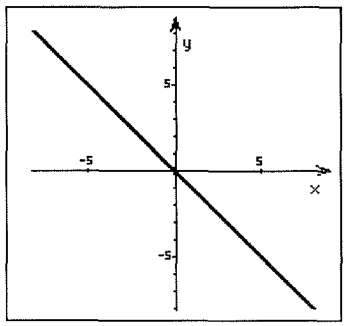


Grafico G

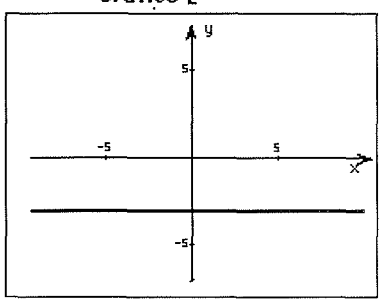


Grafico H

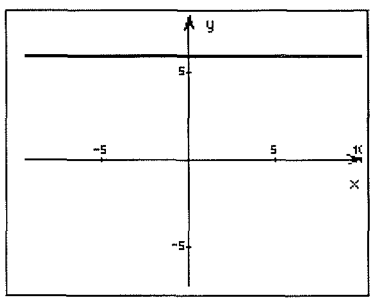


Grafico C

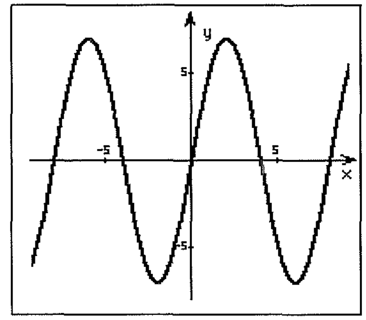


Grafico K

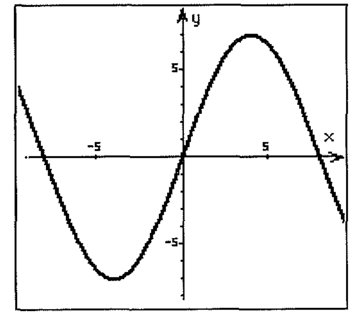


Grafico J

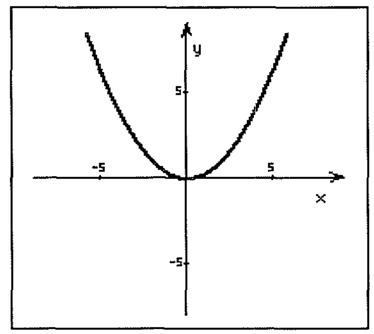


Grafico M

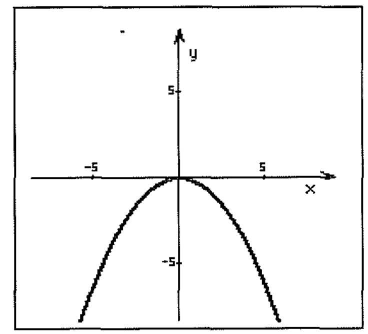


Grafico F

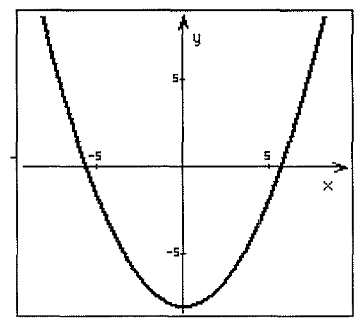


Grafico I

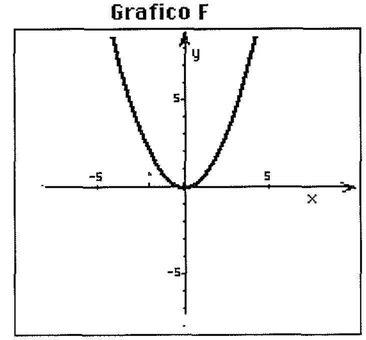


Grafico D

5.4 Cards with the equations

$\boxed{y=-3}$ $\boxed{y=2x}$ $\boxed{y=6}$

$\boxed{y=-x}$ $\boxed{y=0.25x^2-8}$ $\boxed{y=x-6}$

$\boxed{y=x}$ $\boxed{y=0.5x^2}$ $\boxed{y=0.25x^2}$

$\boxed{y=-0.25x^2}$

$\boxed{y=7\sin(0.125\pi x)}$ $\boxed{y=7\sin(0.25\pi x)}$

6 Worksheets and cards of DG microworlds

6.1 Worksheets

Worksheet 12

Description

Striker

Behaviour of the striker

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Worksheet 13

Description of groups

Group

Similarities

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Differences

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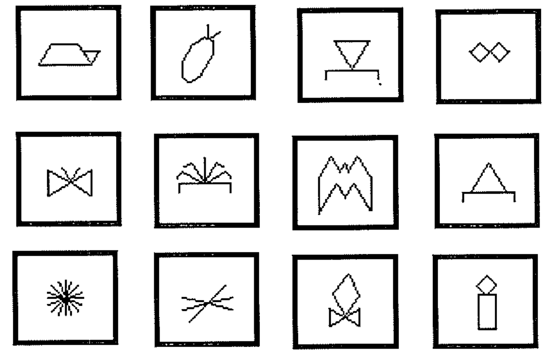
6.2 Cards with the icons of the strikers



7 The final interview - an example

The final interview with John & Tanya

First of all, I gave to the students the cards of strikers and the cards of graphs. Also, I made available to them the horizontal version of DG. Then, I requested the students to correspond the strikers' behaviour to the graphs. With this activity, I intended to see which properties they correlated the behaviour of the strikers to the graphic features while trying to match the strikers to the graphs.

From DG to FP

In the meeting to verify the connections made by the students from DG to FP, I used the horizontal version of DG to remind and show them the points and the cards of graphs for them to explain their answers.

Variation

Constant

Remind them that there are two strikers (which they had used) with a motionless behaviour. Also, that they observed that y doesn't move while x does.

QUESTION(S): How do you recognise 'the motionless behaviour' from graphs?

Monotonicity

Show that some strikers follow the same orientation as the triangle and others don't [$y=x$, $y=-x$, $y=0.25x^2$].

QUESTION(S): How can you recognise if a striker will follow or not by the graph? And How do you know by the graph if a striker will change orientation or not?

Speed

Remind them that among the strikers corresponding to linear functions they discriminated the striker for which y is quicker than x, and for which y has the same speed as x.

QUESTION(S): How do you know the speed of the y and x from graphs?

Do the same with strikers corresponding to parabolas and sines.

Remind them that they recognised in a striker corresponding to a linear function a proportion in its motion while in the strikers given by parabolas they didn't.

QUESTION(S): Is it a property of linear strikers? How can you discriminate if there is a proportion or not by the graph?

Show them that the striker of $y=x-6$ comes behind the striker of $y=x$ with same proportion (as they said: it is a couple from the country, "the wife always comes behind the husband but following her husband").

QUESTION(S): What does it means in your graphs?

Turning point

Remind them that some strikers change orientation, I mean, sometimes they follow the triangle sometimes not. Also that the striker given by $y=0.25x^2-8$ changes orientation at a different point (not on zero).

QUESTION(S): What does it mean in a graph? How can you recognise the change of orientation of a striker by the graph?

Show them that the strikers of $y=0.25x^2$ and $y=0.25x^2-8$ move to 0 and -8, respectively. Also, that the striker corresponding to $y=-0.25x^2$ comes from the other side up to 0.

QUESTION(S): What does it mean to the graph?

Bounded/Boundless

Remind them that the striker goes to infinity positive or to infinity negative.

QUESTION(S): How can you recognise by the graph if the striker will go to infinity or not?

Periodicity

Remind them that they observed that the path of y of the strikers given by sines repeats continually.

QUESTION(S): How can you discriminate it in a graph?

Remind them that they counted how far x moves while y completes a cycle. Also, that they said this is the period of the striker. Also, show that those two strikers have different periods (as they said).

QUESTION(S): How can you illustrate the time that x takes in a graph?

Range

Show them the strikers given by y=7sin(0.25πx), y=x, y=0.25x^2. Remind them that they said some of them move from -8 to 8, another comes from infinity to infinity negative, another from infinity negative to zero, and so on.

QUESTION(S): How do you know by the graph the place where the striker will be able to stay? And How do you know by the graph if it will go to infinity?

From FP to DG

For this part of the meeting I used FP software to trace the graph and to promote transformations in the graphs. I asked them to guess the striker which corresponds to the transformed graph and how they can recognise the graphic feature in the strikers. After answering each question, the students used DynaGraph to verify their answers.

Range

Remind them that each graph can stay in different parts of the y-axis:

QUESTION(S): How can you recognise it in the behaviour of a striker?

Remind them that they talked about dimension of range, for example: in y=-3 a point, in the parabolic one side, all the axis to linear functions, and the interval in sines.

QUESTION(S): How can you recognise the range of a function by the strikers?

QUESTION(S): How can you recognise the dimension of range in the strikers?

Show them that you can promote a vertical translation in the graph of a sine without changing the dimension, and a vertical stretch, changing the dimension of range.

QUESTION(S): Can you predict the behaviour of the new strikers?

Do the same to a parabola:

Boundless/Bounded

Remind them that they talked about the graphs being boundless or not. Show the graph of linear function, in which they said that y is from infinity negative to infinity positive.

QUESTION(S): How can you recognise that a function goes infinity positive or negative in a striker?

Remind them that they said that a graph with cup-shape (parabolas or sines) has minimum.

QUESTION(S): How can you recognise in a striker if a function has minimum? Which of the strikers have minimum?

Do the same to maximum

Promote a vertical translation in a parabola.

QUESTION(S): What will happen to the minimum of the new striker?

Variation

Constant

Show them that some of the graphs we worked with were straight lines parallel to the x-axis.

QUESTION(S): How can you identify if a striker will be a straight line parallel to the x-axis?

Monotonicity

Show to them graphs of parabola, linear, sine functions and discuss with them what is progressive or regressive.

QUESTION(S): How can you recognise if a striker is progressive or regressive? Or where a striker is progressive and where it is regressive?

Speed

Remind them that the curvature of a parabola can change.

QUESTION(S): What will change in a striker if I change the curvature of a parabola?

Show them that in graphs of linear functions there are different slopes (y=x, y=2x, y=x-6).

QUESTION(S): How can you discriminate the slope of a function in a striker?

Also, Remind them that y=x, y=x-6, and y=-x form the same angle with the x-axis .

QUESTION(S): How do you know if two strikers correspond to straight lines with same angle?

Show them that there are curved and straight graphs.

QUESTION(S): How can you discriminate if a striker represents a straight line or non-straight line?

Compare the graph of y=x and y=0.25x^2 in the positive side only. Hide the equation and ask if they can distinguish the strikers in the positive domain.

Remind them that there are up and down concavity parabolic functions.

QUESTION(S): How can you recognise if a striker corresponds to a cup-shape or hill-shape parabola?

Promote a vertical reflection in $y=0.25x^2$

QUESTION(S): How will the new striker be?

Do the same question to sins.

Symmetry

Remind them that a parabola is a symmetric graph, talk about line of symmetry.

QUESTION(S): What does it mean in a striker that its corresponding graph has a line of symmetry?

Promote a horizontal translation in the parabola in order to change its line of symmetry.

QUESTION(S): What will happen to the striker?

Periodicity

Remind them that they said that those graphs of sins were periodic.

QUESTION(S): How can you discriminate a periodic striker?

Remind them that they had affirmed that the sines graphs had periods of 8 and 16.

QUESTION(S): How can you recognise if a striker corresponds to a periodic function? And how can we compute the period of a periodic striker?

Promote a vertical stretch in the graph of a sin.

QUESTION(S): What will happen to the new striker?

Promote a horizontal stretch in the graph of a sin.

QUESTION(S): What will happen to the new striker?

Remind them that in the graph of sins they said that the turning points were periodic.

QUESTION(S): How can you recognise the periodicity of a turning point in a striker?

Turning point

Remind them that some graphs have turning point(s), also that they affirmed that 'a turning point is where a graph changes from increasing to decreasing or vice-versa', as well as the "apice" of a graph.

QUESTION(S): How can you recognise the turning point in a striker?

Promote a vertical translation in a parabola as well as a horizontal translation.

QUESTION(S): How will the new striker be?

Show to them that they were localising the value of a turning point in a graph.

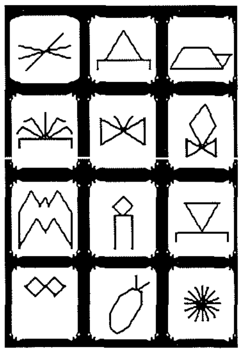
QUESTION(S): Could you do it in a striker? How?

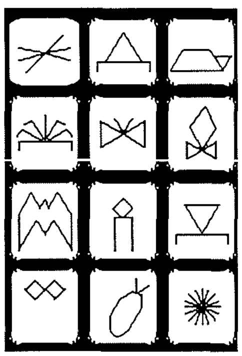
8 Sheets for helping the observations

8.1 Observations for DG microworlds

Notepad 1

Pair of Students:________________ Session: ________





8.2 Observation sheet for FP

Notepad 2

Pair of Students: ______________ Session: ______________

	-3	6
	x	$x-6$
	$2x$	$-x$
	$\frac{1}{4}x^2-8$	$\frac{1}{2}x^2$
	$\frac{1}{4}x^2$	$7\sin(\frac{\pi}{8}x)$
	$-\frac{1}{4}x^2$	$7\sin(\frac{\pi}{4}x)$

	-3	6
	x	$x-6$
	$2x$	$-x$
	$\frac{1}{4}x^2-8$	$\frac{1}{2}x^2$
	$\frac{1}{4}x^2$	$7\sin(\frac{\pi}{8}x)$
	$-\frac{1}{4}x^2$	$7\sin(\frac{\pi}{4}x)$

8.3 Observations of students' perceptions built within DG microworlds

Notepad 3

Observations for synthesis

DG Parallel and DG Cartesian

Pair of students: ______________________________

Perceptions	Ses.	Observations
Change in the orientation of the motion.		
Constant speed		
Different value of speed.		
Distance between x and y increases proportionally		
Distance between x and y keeps the same		
Range - place where y can stay.		
in a line		
in a line		
in a point		
in a semi-line		
Orientation of the motion.		
Periodic passage of the striker by a value.		
Periodicity - path that y repeats		
Symmetry of x to y		
Variable speed		
Turning point - Bounded motion of the striker		
Turning point - limit of motion of y		
y and x move with fixed ratio		
y and x move with same step		
y doesn't move while x does		
y is always positive or negative		
y moves in interval		
y tends to infinity		

8.4 Observations of students' perceptions built within FP

Notepad 4

Observations for synthesis

Function Probe

Pair of students:______________________________

Perceptions	Ses.	Observations
Amplitude		
Compare curvatures		
y does not depend on x		
Even or odd function		
Range		
Monotonicity		
Line of symmetry		
Linearity		
Extreme values		
Parallel straight lines		
Period (value)		
Periodicity of roots		
Periodicity of turning points		
Periodic function		
Positive and negative velocity		
Slope (different slopes)		
Slope - negative/positive		
Straight line		
Straight line parallel to x-axis or not		
Symmetry on (0,0) or x-axis		
Boundless/bounded range		
Positive or negative curvature		
Turning point - Change from increasing to decreasing		
Turning point - value		
y tends to infinity		

Appendix II - Details of the Activities

1 Starting activities with DG Parallel

The starting activities in DG Parallel were designed to familiarise students with its structure. The students were request to play with a computer-game, called DG Game, as a first contact with the DynaGraph dynamic way of representing functions. In a second phase, they would use DG to describe the behaviour of all the strikers.

DG Game is a computer-game adapted from DG Parallel. It differs from DG Parallel by: (a) the number of strikers which can be active; DG Game allows only one active striker and (b) the inclusion of game features. A ball was included in the same line as the active striker. This ball changes place randomly. In order to score the students make the strikers strike the ball. As described in the section on DG Parallel software, the striker moves according to the motions promoted in the triangle (x) and the function hidden in the active striker by the students.

Figure 1.1

Screen of DG Game

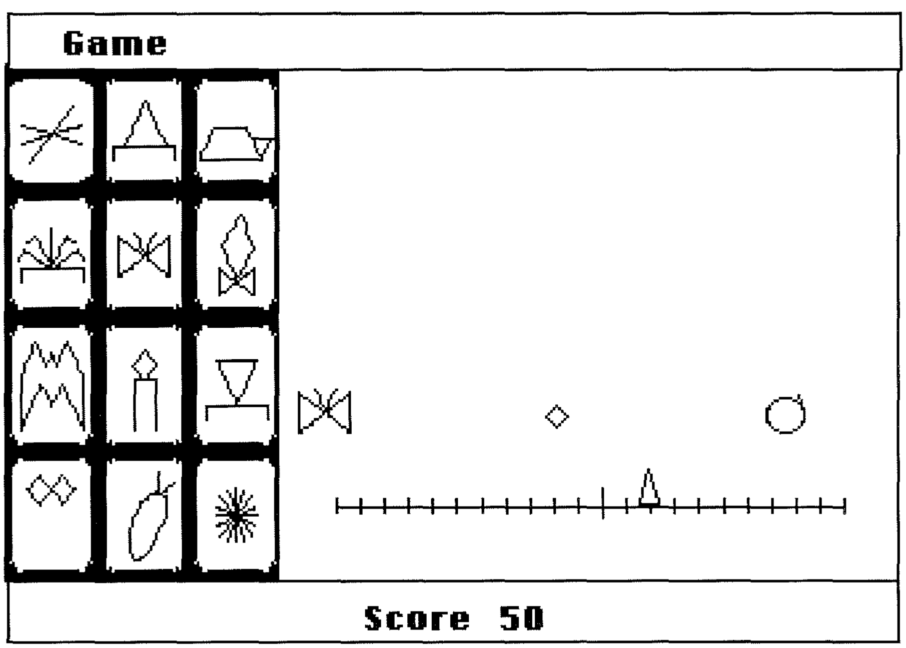


Table 1.1 summarises the activity with DG Game showing: step-by-step description of the activities proposed to the students, aims for the students, and justifications and/or aims for the research.

Table 1.1
Description and aims of the starting activities with DG Parallel

Step-by-step	Aim for the students	Aim for the research
Students receive information on how to operate DG Game and are asked to try each striker	Play with DG Game	Familiarise students with the structure of all the versions of DG
Students are asked to play with DG Game (without being informed that they are dealing with functions), meanwhile the students are encouraged to talk about the different strikers	Explore the behaviour of the twelve strikers in a game environment.	Motivate the students to freely explore the functions as they follow the behaviour of the strikers Promote discussion about different behaviour of the strikers

2 Activities around DG Parallel and DG Cartesian

The activities in DG Parallel and in DG Cartesian will be detailed together because they have a common structure (see table III-4.2). Whenever 'the software' is used, it will mean DG Parallel or DG Cartesian depending on the microworld of the activity. Table 2.1 shows the activities from the session on description and table 2.2 those from the session on classification with aims for the students and justification for the research.

Table 2.1
Details of activities of description in DG Parallel and DG Cartesian

Act. N.	Step-by-step	Aims for the students	Justification
Previo us	Students are informed how the software represents a function	Know how they will be working with functions	Allow them to continue building situa ted abstractions as well as to conne ct these with previous knowledge
DG Par (2) (3) DG Cart (1) (2)	Students repeat describing/guessing activity using two strikers each time. The next three steps are repeated until they describe 8 strikers	Search for characteristics which can describe and distinguish the strikers.	Lead them to compare properties between functions Lead them to search for new properties in one function (in the case of the two descriptions being the same)
	One student leaves the room while the other stays to describe two strikers	Participate in a describing/guessi ng activity	Enable the describing/guessing activity to take place
DG Par (2) DG Cart (1)	The student chooses two strikers, tries them (and others if s/he wishes to) in the software and writes down one description for each in worksheet 12	Characterise the strikers according to aspects they perceived in the behaviour of the strikers.	Lead them to search for properties of functions in each striker. Lead them to compare properties between different functions Apart from the initial descriptions, lead them to generalise properties earlier discriminated by themselves
DG Par (3) DG Cart (2)	The partner comes back, guesses the striker from the description. S/he tries the strikers in the software. Both students stay in the room. The first student is allowed to defend his/her own description	Compare the description made by the partner with the behaviour of the strikers in the software	Lead them to compare the properties of different functions Lead each of them to review his/her own perception of the properties considering the ones from his/her partner. Lead each of them to discuss his/her perceptions of the properties with his/her partner Lead them to check the accuracy of the description when finding out whether two strikers or none could match the description
DG Par (2) DG Cart (1)	Together students describe the remaining 4 strikers while trying the strikers in the software	Discuss their perceptions in order to reach a common description for each striker	Lead them to conclude the description of all the functions, not with guesswork at this stage Lead them to review and generalise their perceptions of the properties by discussion Lead them to search for more properties in the same function
Note	In all the steps above, students can freely explore one, two or three strikers in the screen	Compare behavio ur of different strikers as well as concentrate on only one	Allow them to search for new properties (or aspects of the properties) comparing similarities and differences among functions

Table 2.2 refers to the activity (4) of DG Parallel and (3) of DG Cartesian

Table 2.2

Details of activities of classification in DG Parallel and DG Cartesian

Step-by-step	Aims for the students	Justification
The students are given their own descriptions and are asked to group the strikers according to their common characteristics. They use the 12 cards to group them on the A3 paper	Negotiate a way to group the strikers	Allow them to compare their perceptions of the properties among all the functions Lead them to generalise properties of function by identifying similarities and differences of the properties among different functions
Students write down a description for each group of strikers using the worksheet 13, the description contains similarities and differences within and between the groups	Write down the similarities and differences which led them to decide the groups	Lead the students to conclude the generalisation of their perceptions of the properties between and within groups of functions.
The students check their descriptions by exploring the strikers in the software, in general comparing different strikers at the screen. The students are allowed to change the classification they made themselves.	Check whether similarities and differences they considered in the descriptions of the group are actually perceived in each striker. Rebuild the groups	Lead them to review their generalisations by trying the strikers again Lead them to compare properties between functions classified within and between groups. Lead them to compare functions with in and between families of functions

The activities in DG Parallel and DG Cartesian microworlds are justified above. The sequence from one to the other is designed to direct the perceptions which the students derived from DG Parallel into the Cartesian system. Thus, an additional justification for the activities of DG Cartesian is to lead the students to use their perceptions of the properties derived from activities in DG Parallel as well as those derived from previous knowledge about Cartesian systems.

3 Starting activities with FP

The main purpose of the starting activities in FP is to familiarise students with the use of transformations on graphs. The function given by y=abs(x) was chosen.

Table 3.1
Details of the starting activities with FP

Act. N.	Step-by-step	Aims for the students	Justification
	FP is available The students receive a notepad with the equation y=abs(x) (worksheet 1)		
Previo us	The students are asked to participate in the describing/guessing activities using one of the transformations on the graph of y=abs(x). The next three steps are done for each of the transformations changing the role of the students	Being introduced to the type of activity	Familiarise students with the describing/guessing activity which will be used in the other sessions
(1) Previo us	One student leaves the room The other student traces the graph of y=abs(x) in FP and chooses one command to transform it	Obtain the graph to start the transformation	Enable the describing/ guessing activity to take place Familiarise students with the operation of each transformation command
(1)	The student transforms the graph obtaining a second graph, taking notes of the new equation in the notepad (Worksheet 1)	Explore the chosen transformation in the graph of y=abs(x) to obtain a new graph	Familiarise the student with the operation of the chosen transformation on graphs
(2)	The student describes the graphs obtained and also adds the similari ties and differences between the two graphs (worksheet 2)	Describe the graph obtained	Familiarise the student with the 'describing' activity that will be used in the other sessions
(3) Previo us	The second student retu rns to the room, receiv ing the description only		Enable the describing/guessing activity to take place
(3)	The second student explores the chosen transformation to obtain the described graph	Guess the described graph	Familiarise the second student with the operation of the chosen transformation Familiarise the students with the 'guessing' activity

4 Activities around FP software

As activities were designed exploring only the Graph window of FP, to detail the activities I will use FP meaning the software with only the Graph window displayed.

Table 4.1
Details of activities of description in FP (continues in table 4.2)

Act. N.	Step-by-step	Aims for the students	Justification
Previous	The students are asked to participate in repetitive describing/guessing activities using two different ways of describing: (a) with worksheet 3 by comparing two graphs or (b) with worksheet 4 by exploring all the dynamic transformations in one graph. They are informed that they can freely choose which actitivity they will do	Know the structure of each of the activities Feel free in their choice	Familiarise students with the worksheets Give them freedom to follow their own paths while exploring the properties of functions in the software
(1) Previous	One student leaves the room FP is available The other student receives the set of cards with equations. S/he chooses which worksheet s/he will use	Choose the type of activity s/he desires to do in the turn	Enable the describing/guessing activity to take place Give the students a way to input the graphs into the software Allow them to follow their own path while exploring the properties of functions
(1a)	The student chooses two equations and traces one of them in FP	Choose the functions s/he will explore	Give him/her freedom to choose the functions s/he wants to explore
(2a)	The student tries to obtain the graph of the second equation using the transformations of graphs from FP	Explore the dynamic transformations of the graphs to describe the second graph	Lead him/her to discriminate properties of function as variants or invariants of the transformations of graphs Lead him/her to perceive new properties. Lead him/her to review his/her previous perceptions
	The student writes down: the equation of the first graph and a description of the second graph including similarities and differences between the two graphs using worksheet 3	Characterise the second graph by comparing it with the first one	Lead him/her to resume variant and invariant properties perceived while transforming graphs.

Table 4.2
Continuation of table 4.1 (Details of activities of description in FP)

Act. N.	Step-by-step	Aims for the students	Justification
(1b)	The student chooses one equation and traces it in FP	Choose the function s/he will explore	Let him/her follow the most appropriate path to review his/her perceptions by his/her own opinion
(2b)	The student explores one-by-one the dynamic transformations in the graph traced, taking notes of the variants and invariants of the process	Explore the dynamic transformations in the graph to describe it	Lead him/her to perceive new properties as variants and invariants of the transformations. Lead him/her to review previous perceptions of the properties exploring the transformations of the graph Lead him/her to generalise properties by perceiving them as invariant among families of functions
	The student writes down a description of the graph using worksheet 4	Describe the graph with the properties observed as variants and invariants of the transformations	Lead him/her to resume variant and invariant properties s/he perceived while transforming the graph
(3) Previo us	The student gives back the cards with equati ons to the researcher The second student re turns to the room and receives the descrip tion made by his/her partner and the set of cards with graphs		Lead the first student to write only characteristics of graphs in the worksheets Do not allow the second student to use the equation to guess the function Direct the study to investigation on graphs
(3)	The second student tr ies to match the descri ption with one of the graphs from the cards. Students are encouraged to discuss	Guess the graph described by his/her partner	Lead them to review their perceptions of the properties by comparing with their partner's perceptions Lead them to check the accuracy of their descriptions if none or more than one of the graphs can be fitted in the description Lead them to search for new properties of functions
	The students are allowed to use FP in the case of not having agreed about the graph described	Negotiate agreement on the description by using the transformations to review it	Lead them to search for new properties as a way to distinguish two graphs while investigating the transformations Lead them to review their perceptions in the light of partner's perceptions.

Table 4.3

Details of activities of classification in FP

Act. N.	Step-by-step	Aims for the students	Justification
(1)	The students receive the cards of graphs and the descriptions they themselves made They are asked to group the graphs according to their descriptions on A3 paper	Classify the graphs according to the properties identified	Lead them to generalise the properties among similar graphs Lead them to compare their perceptions by negotiating a common classification Lead them to compare their perceptions of the properties within and between families of functions
(2)	FP is available to the students The student chooses one graph of each gro up to explore the dyna mic transformations on these graphs using worksheets 6 to 9 The students are allowed to rebuild their classification	Explore the effects of dynamic transformations on graphs of different groups	Lead them to refine their perceptions of the properties by investigating the transformations on different graphs Lead them to compare their perceptions of the properties by exploring one transformation on graphs of different groups Lead them to generalise properties by perceiving them as invariant among families of functions
(3)	The students write down one description for each group in worksheet 5 including common and variable properties of the graphs in the group	Describe the variants and invariants used by them to classify the graphs in groups	Lead them to conclude their perceptions of the properties within and between groups, thus within and between families of functions

Appendix III — An Example of the Reports of the Data

This appendix intends to support the discussion of the process of building the blob diagrams in chapter IV. It will take as example the analysis of Bernard & Charles while they investigated constant function. Thus, all the sections below, apart from the tables, will refer only to those parts of the report which discuss their perceptions of constant function.

1 The pre-test

Bernard & Charles had no ability in sketching graphs from equation. Bernard used the method of plotting graphs by: calculating a few points from the equations, plotting these points, and linking them. Nonetheless, he changed the x-axis and the y-axis. Thus, the graph of y=2 was traced as a vertical straight line. As far as the graph traced by the verbal description is concerned, only Charles tried to sketch it. He sketched a motionless car in the graph of distance per time as a dot.

Bernard & Charles presented a change between the meaning of the terms constant and periodic. Bernard drew a horizontal straight line to show what he understood by periodic function and a graph of a sine to show what he understood as a constant function. Charles answered that a constant function is a function whose graphs have a repetitive path. Nonetheless, when asked to identify the periodic graphs, he behaved in two ways: graphs traced by himself were interpreted to be periodic when they were straight lines; graphs given in the pre-test were interpreted to be periodic in the original meaning.

2 DG Parallel

After recording and transcribing each session with a pair of students, the video-tapes were watched to complete, check and analyse the transcription. Then, I analysed and constructed a table to summarise their findings in each of the properties. Here, I will present the analysis of the findings regarding constant function in DG Parallel.

Codes used in tables 2.1, 3.1, 3.2, 4.1 and 4.2:

For the properties:	For the phases of perceptions:	For the commands of FP:
TP - Turning point	(D) - Discriminate	[VT] - Vertical translation
CF - Constant function	(G) - Generalise	[HT] - Horizontal translation
MT - Monotonicity	(A) - Associate	[VS] - Vertical stretch
DR - Derivative	(S) - Synthese	[HS] - Horizontal stretch
SD - Second derivative	(L) - Linked to	[VR] - Vertical reflection
RG - Range	(R) - Recognised by	[HR] - Horizontal reflection
SM - Symmetry		
PD - Periodicity		
SP - Shape		
PT - Point		

Table 2.1
Summary of Bernard & Charles' findings in DG Parallel

Function	Prop.	How it is identified	Classific.
y=6	CF (D)	Motionless striker	CF (Motionless strikers)
	CF (D)	'It is useless to move x, the striker doesn't do anything'	
y=-3	CF (D)	Motionless striker	
	RG (D)	The striker stays at the right side	
y=x	MT (D)	The striker follows the triangle (x)	
	DR (G)	The striker has the same speed as the triangle (x)	
	RG (G)	The striker moves all the screen	
	SD (G)	Constant speed (A) x and y have the same speed	
y=-x	MT (D)	The striker moves in the opposite orientation of the triangle (x)	RG (The striker moves all the screen)
	DR (D)	The striker moves with the same speed as the triangle (A) the striker has the same distance to zero as the triangle to zero	
	PT (D)	y and x meet at zero	
	SM (D)	y is symmetric to x	MT (The striker moves in only one orientation)
y=x-6	DR (D)	y has the same speed as x	
	RG (G)	The striker moves all the screen	
y=2x	MT (D)	y and x move in the same orientation	
	RG (G)	y moves all the screen	
	DR (D)	The striker is quicker than the triangle	
	SD (D)	The striker slows down (A) it starts ahead of the one of $y=0.25x^2$ and then is overtaken	
$y=0.25x^2$	RG (D)	The striker moves only in the right side	RG (The striker only goes to one end of the screen)
	DR (G)	The striker is quicker than x	
	MT (G)	Sometimes the striker follows x, sometimes it does not	
	TP (G)	The striker changes orientation at zero (x and y are zero)	
	SD (G)	The striker speeds up (A) it starts behind the one of y=2x and then overtakes it	
$y=-0.25x^2$	RG (D)	The striker moves only in the negative side	
	RG (G)	The striker only moves in one side	
	TP (G)	The striker changes orientation at zero	
	DR (D)	The striker is quicker than x	
$y=0.5x^2$	DR (D)	y is quicker than x	
	RG (D)	The striker does not move in the negative side	
	RG (G)	The striker moves only in one side	
$y=0.25x^2-8$	RG (D)	The striker returns when it arrives at -8	
	RG (D)	The striker does not go to the end of the axis	
	DR (G)	Striker is quicker than the triangle	
	TP (D)	y returns at -8 when x is in the middle (at zero)	
	RG (G)	Striker does not go to one end of the screen	
	SD (D)	The striker stops near the turning point	
$y=7\mathrm{Sin}(0.25\pi x)$	TP (D)	y returns when it arrives at a limit value	RG (the striker moves only in the middle of the screen)
	RG (D)	y returns when it arrives at a limit value	
$y=7\mathrm{Sin}(0.125\pi x)$	RG (G)	The striker does not go to the end of the screen (both sides)	
	MT (G)	The striker does not obey x	
	DR (D)	The striker is quicker than the triangle	

The strikers corresponding to constant functions were characterised by this pair of students as motionless. Thus, these strikers were considered to be completely different to the strikers corresponding to linear functions. Bernard & Charles' characterisation of y=6 reflected the idea that 'y is independent of x'. They affirmed that it was a nonsense striker — "It is useless to move it [x], it [the striker] doesn't do anything".

In DG Parallel version Bernard & Charles classified the strikers of constant functions using the motionless criterion. It is interesting that, in this representation, these strikers were considered by the students to be completely different from the strikers of linear functions.

3 DG Cartesian

Table 3.1

Summary of Bernard & Charles' findings in DG Cartesian

Function	Prop.	How it is identified	Classific.
y=6	CF (S)	y is motionless (L) y is constant	SP (horizontal straight line) (R) CF (y is constant)
	PT (G)	(x,y) does not pass through the origin	
	CF (S)	(x,y) moves in horizontal straight line (L) y is constant	
y=-3	CF (D)	y is motionless	
y=x	MT (D)	The straight line grows to the right side	SP (Diagonal straight line) (R) RG (y moves all the axis)
	DR (G)	The striker has same speed as x	
	DR (G)	Proportion 1 to 1	
	SD (G)	Proportion 1 to 1	
	RG (G)	The striker moves all the graph	
	PT (D)	The figures [x,y,(x,y)] meet each other in the middle [(0,0)]	
	SP (D)	(x,y) describes a straight line	
y=-x	MT (D)	Direction of the straight line (A) the straight line is positive to the left side	
	DR (D)	The striker has same speed as the triangle (x)	
	SD (D)	The striker has same speed as the triangle (x)	
	RG (G)	The striker moves all the y-axis	
	PT (D)	The figures [x,y,(x,y)] meet each other in the middle [(0,0)]	
	SP (D)	(x,y) describes a straight line (R) The dot (x,y) does not make a turning point	
y=x-6	MT (D)	The term 'increasing' (A) the straight line is positive after 6	
	DR (G)	The striker has same speed as x	
	SD (G)	The striker has same speed as x	
	RG (G)	The striker moves all the y-axis	
	PT (D)	The dot [(x,y)] meets x at 6.	
	SP (D)	Straight line (R) the dot does not make a turning point	
y=2x	DR (G)	The striker is quicker than x	
	DR (G)	Proportion 2 to 1	
	SD (G)	Proportion 2 to 1	
	RG (G)	The striker moves all the y-axis	
	PT(D)	The figures [x,y,(x,y)] meet in the middle [(0,0)]	
	SP (D)	(x,y) moves in a straight line	

Table 3.2

Continuation of the summary of Bernard & Charles' findings in DG Cartesian

Function	Prop.	How it is identified	Classific.
$y=0.25x^2$	RG (D)	The striker moves only to the middle of the screen	SP (Parabola) (R) TP ((x,y) changes orientation once) RG (Infinity to one side)
	SD (G)	There is no fixed proportion between x and y (L) term 'irregular' proportion	
	SP (D)	(x,y) forms a parabola	
$y=-0.25x^2$	RG (D)	y moves only in the negative numbers	
	TP (D)	(x,y) changes direction at (0,0).	
$y=0.5x^2$	DR (D)	y is quicker than x	
	RG (G)	y moves only in the positive part of the y-axis	
$y=0.25x^2-8$	RG (D)	The striker moves all the axis (positive and negative)	
	SD (D)	There is no fixed proportion	
	TP (G)	(x,y) changes direction at (0,-8)	
$y=7Sin(0.25\pi x)$	SP (D)	Constant parabolas (R) curve with a turning point that repeats	SP (Constant Parabolas) (R) (y keeps repeating the same path and it keeps changing orientation) RG (Its beginning and end can be marked)
	DR (D)	The striker is quicker than the striker of other sine	
	RG (D)	The striker moves half of the axis	
	PD (D)	The frequency of turning points is bigger than in the other striker of sine	
$y=7Sin(0.125\pi x)$	SP (D)	Constant parabolas (R) curve with a turning point that repeats	
	DR (D)	The striker is slower than the striker of other sine	
	PD (D)	The frequency of turning points is smaller than in the other striker of sine	

In the first analysis of the striker given by y=6, Bernard confused the idea 'y is motionless' from DG Parallel with the idea '(x,y) is motionless'. He argued that this striker had the same speed as x while observing the sprite corresponding to (x,y). Only when he noticed that 'y was motionless', did he argue that as 'y is constant', the '(x,y) moved in a horizontal straight line'.

In DG Cartesian Bernard & Charles used family of functions to classify the strikers. These families were characterised by the shape (x,y) traced on the screen. Nonetheless, it is important to notice which ideas were used to recognise these shapes. For straight lines, they used the range 'y moves all the y-axis' to recognise their family. First, the strikers of y=x-6, y=x, y=-x were classified in the same group because of the above-mentioned characteristic plus the fact that they all have the same speed as x. Before that, the strikers of constant functions were characterised as horizontal straight lines, which was recognised by y is constant. Consequent to that, the shape traced by (x,y) of the strikers of linear functions were distinguished as diagonal straight lines.

4 FP

Table 4.1
Summary of Bernard & Charles' findings in FP

Function	Prop.	How it is identified	Classific.
y=6	RG (D)	y is positive [VT]	SP (horizontal straight line) (L) CF (y is constant)
	CF (S)	Horizontal straight line (L) y is constant [DG]	
y=-3	RG (D)	y is negative [VT]	
	CF (S)	Horizontal straight line (L) y is constant [DG]	
y=x	MT (G)	Direction of straight line observing different slopes [HS]	
	DR (G)	Angular coefficient (L) Proportion 1 to 1 [HS] [DG]	
	DR (S)	Ratio between absolute values of y and x (L) Inclination [HS]	
	PT (D)	The graph passes through the origin	
	SM (S)	Symmetric to y=-x (L) $f_1(-x)=f_2(x)$ [VS]	
y=-x	MT (D)	Direction of the straight line	SP (Diagonal straight line) (L) DR (y varies)
	PT (D)	The straight line passes through the origin	
	SM (S)	Symmetric to y=-x (L) $f_1(-x)=f_2(x)$ [VS]	
y=x-6	DR (S)	Proportion 1 to 1 (L) Parallelism with y=x	
	DR (S)	Proportion 1 to 1 (L) Compare the behaviour of the strikers of y=x and y=x-6	
	SM (D)	The graph crosses the x-axis and the y-axis in symmetric values	
y=2x	MT (D)	Direction of the straight line	
	DR (S)	Proportion 2 to 1 [DG] (L) Linear coefficient [VT]	
	DR (S)	2 to 1 is the symmetric of the ratio between y-intercept and x-intercept [VT]	
	PT (D)	The graph passes through the origin	
$y=0.25x^2$	TP (G)	The value of y [VT]	
	SD (D)	Same curvature of the graph of $y=0.25x^2-8$ [VT]	
	SD (S)	Curvature (L) absence of proportion	
	SP (D)	Parabola	
$y=-0.25x^2$	TP (G)	Point where the graph changes direction [HS]	
	SM (G)	Line symmetry in the y-axis	
	SP (D)	Parabola	
$y=0.5x^2$	TP (D)	At (0,0) [VT]	SP (Parabola)
	SD (D)	Compared with the curvature of the graph of $y=0.5(x/(-7))^2$ [HS] and $y=0.5x^2-10$ [VT]	
	SD (S)	Curvature (L) absence of proportion	
	SP (D)	Parabola	
	SM (G)	Line symmetric graph (A) $f(x)$ must correspond to $f(-x)$	
	SM (G)	Identification of the line of symmetry in the y-axis [HT]	
	RG (D)	y is only positive	
$y=0.25x^2-8$	TP (D)	The value of y is -8 [VT]	
	SD (D)	Same curvature of the graph of $y=0.25x^2$ [VT]	
	SP (D)	Parabola	
	PT (D)	The roots	

Table 4.2
Continuation of the summary of Bernard & Charles' findings in FP

Function	Prop.	How it is identified	Classific.
y=7Sin(0.25πx)	SP (D)	Repetition of parabolas	
	PD (D)	Distance between two roots [HT] / It repeats from 8 to 8 units	
	PD (G)	Invariant period calculated in turning points of -7 [HS in y=7sin(0.125πx)]	
y=7Sin(0.125πx)	SP (D)	Repetition of parabolas	SP (Continuous Parabolas)
	RG (D)	Interval that its graph describes [VS]	
	PD (S)	The turning point of -7 repeats from 16 to 16 (L) the term period	
	PT (D)	The graphs through the origin	
	TP (D)	Distinction between bottom and top turning points with respective values of y [VT to y=7sin(0.25πx)+6.9]	
	RG (D)	Amplitude (R) height of turning points [HS]	

The graphs of constant functions were described by Bernard & Charles as horizontal straight lines. They stated that the lines were horizontal because y was constant. In my view this can be considered as a synthesis between the idea built within DG microworlds and their knowledge of graph. In their previous knowledge a motionless car was represented by Charles as just one point in the graph. Moreover, the word 'constant' was used by both of them as a periodic function.

When they had the graphs available, the criterion used by Bernard & Charles to classify them was the shape of the graphs. The functions were separated into the following shapes: parabola, 'continuous parabolas', horizontal straight line and diagonal straight lines. The slope of the straight lines was used by Bernard & Charles to separate the functions in which 'y varies' from those functions in which 'y is constant' as explained by Bernard: "in these straight lines [diagonal], y goes changing". This is an influence of the activities with DG microworlds.

5 Analysis of the blob diagram

After building a blob diagram for each of the properties, for example diagram CIV-7.5, I wrote an analysis showing the visual findings prompted by the blob diagram (without the final interview displayed). Here I will present the analysis of the blob diagram of Bernard & Charles' perceptions of constant function as well as the facts which gave evidence for the links.

Evidence for the links:

(A) This is the way Charles and Bernard defined the term constant function;

(B) Charles represented the motionless behaviour of a car in a graph of distance per time as a dot;

(C) The students answered the questions about the meaning of periodic function by drawing a horizontal straight line in the Cartesian system;

(D) After characterising the striker of a constant function as being motionless, they complained about the independence of y from x;

(E,G) During the above-mentioned process, the students changed from 'y is motionless' through 'y is constant' to '(x,y) moves in a horizontal straight line';

(F) Knowing from DG Parallel that the striker of y=6 was the motionless one, Bernard expected that (x,y) did not move. Trying to understand why it was not true, he linked 'y is motionless' to '(x,y) moves in a horizontal straight line' directly;

(H) The students reported that the graphs were horizontal straight lines because 'y was constant'.

All Bernard & Charles' previous perceptions of constant function did not correspond to this property from a mathematical viewpoint. Nonetheless, on following the activities from DG to FP, they built a variational view of constant function which ended with a link with its shape in the Cartesian representation.

DG Cartesian was used as a bridge by these students to connect these variational views, built in DG Parallel, to the Cartesian representation in FP. In my opinion, the mixture of DG Parallel with Cartesian representation was what prompted the bridge in DG Cartesian.

The above-mentioned construction of perceptions seems to have mobilised their previous perceptions presented in the pre-test. At the end, they linked 'y is motionless' to 'horizontal straight line'. Then, I expect that the students' previous perceptions as presented in the pre-test had changed, although I have no evidence of this change.

By stressing the motions of x and y of DG Parallel, the students developed a perception of 'y is independent of x' which was presented exclusively in DG Parallel. It seems that this perception was easily built in DG Parallel because in the constant functions, it does not matter what the students do with x, y will stay in the same place.

5 The final interview

As in the research environment, in the final interview Bernard & Charles connected 'the motionless behaviour of the striker' of constant functions to the shape of its graph. Nonetheless, the important point was the functional explanation of this link: "because y does not change". This explanation demonstrates that DG Cartesian really scaffolded the variational perceptions in Cartesian representations of constant functions while serving as a bridge.

Appendix IV — Tables and Diagrams of Students' Perceptions of the Function Properties

Key for abbreviations used in the following tables:

Students' perceptions
(D) - Discrimination
(G) - Generalisation
(S) - Synthesis
(A) - Associated to
(R) - Recognised by

Dynamic commands from FP
[HT] - Horizontal translation
[VT] - Vertical translation
[HS] - Horizontal stretch
[VS] - Vertical stretch
[HR] - Horizontal reflection
[VR] - Vertical reflection

Microworlds
[DG] DG microworlds
[FP] Function Probe microworld
[FI] Final Interview

1 Turning point

Table 1.1
Summary of the perceptions of turning point in each microworld of the pairs of students who began by working with DG

	Bernard & Charles			Anne & Jane		
Function	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**	**FP**
y=2x				There is no variation on the striker orientation (G)		
y=0.25x^2	The striker changes orientation at zero (x and y are zero) (G)		The value of y [VT] (G)	The striker changes orientation at zero (G)	(x,y) is coming down to zero, stops decreasing and starts to increase (D)	Point where the parabola stops growing and starts decreasing (S) Term turning point
y=-0.25x^2	The striker changes orientation at zero (G)	(x,y) changes direction at (0,0) (D)	Point where the graph changes direction [HS](G)	The striker changes orientation at zero (G)	Turning point (A) maximum or minimum of the function (D)	
y=0.5x^2			At (0,0) [VT](D)	The striker changes orientation at zero (D)		
y=0.25x^2-8	y returns at -8 when x is in the middle (at zero) (D)	(x,y) changes direction at (0,-8) (G)	The value of y is -8 [VT] (D)		Parabola with turning point at 8 (R) (x,y) returns at 8 (G)	Term turning point (S) point where the graph stops decreasing and starts increasing
y=7Sin(0.25πx)	y returns when it arrives to a limit value (D)			The striker changes orientation around 6 and -6 [y] (D)	(x,y) arrives to the top and it goes down (S) maximum and minimum	
y=7Sin(0.125πx)			Distinguished in bottom and top turning points with res pective values of y [VT](D)			

Table 1.2

Summary of the perceptions of turning point in each microworld of the pairs of students who began by working with FP

Function	John & Tanya			Diana & Gisele		
	FP	DG Parallel	DG Cartesian	FP	DG Parallel	DG Cartesian
$y=x$					The striker never changes orientation (D)	
$y=0.25x^2$		y goes down to zero (G)				Parabola (R) turning point described by (x,y) (D)
$y=-0.25x^2$	Top of parabola [VT](D)	y goes up to zero (G)		The graph has a point of maximum (D)	Point where the striker returns (D)	
$y=0.5x^2$				The point of minimum of the parabola (D)		Parabola (R) Curve with turning point going up (D) It has minimum (R) turning point with cup-shape (D)
$y=0.25x^2-8$	The value of y is -8 (S) Coefficient of the equation [VT]	The value that bound the motion of the striker in -8 (D)		The graph has a minimum (D) The turning point is at -8 [HT; VR] (D)	y comes up to a point and returns (D)	Parabola (R) Curve with turning point which goes up (D)
$y=7Sin(0.25\pi x)$	Many parabolas (R) Curve with turning point (D)	y arrives to here (-7) and returns (D).	Bound of the motion of y (D) Point where the striker changes orientation (D) The values of y are -7 and 7(D)	The graph has maximum and minimum points (G) Heights of the turning points do not vary (D)		Sine function (R) (x,y) traces many curves (S)
$y=7sin(0.125\pi x)$		y does not overtake -8 and 8 (G)		It has maximum and minimum (G)	Place where the striker changes orientation (D) The striker returns at the same absolute value at -7 and 7 (D)	Sine function (R) (x,y) traces many curves (D)

Diagram 1.1

Comparison of the students' perceptions of turning point

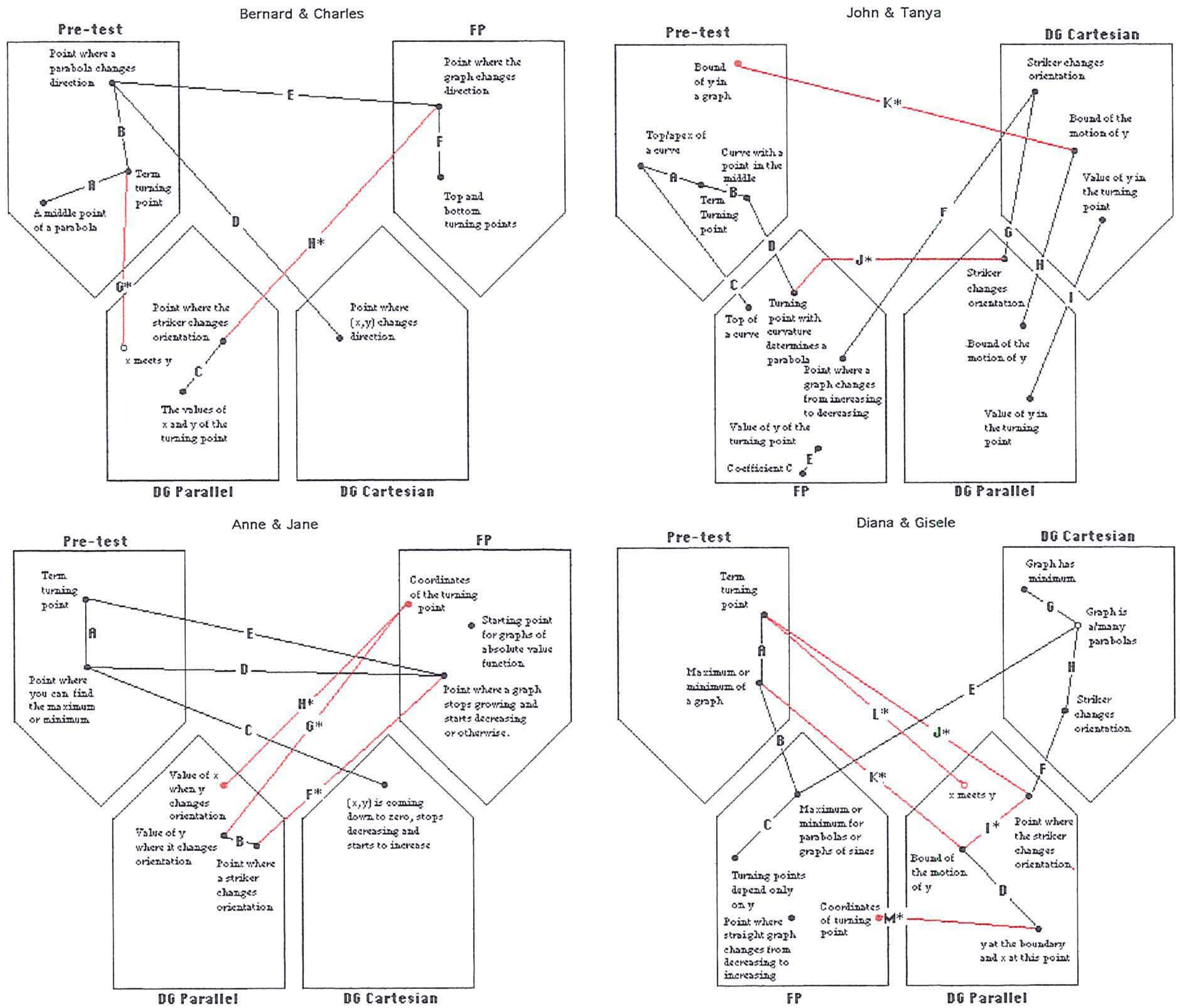


Table 1.3

Changes in students' perceptions of turning point in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Horizontal translation			The point where a graph changes from increasing to decreasing [y=abs(x)]	Turning point does not change [$y=0.25x^2-8$]
Vertical translation	Coordinate [$y=0.5x^2$]		Top of parabola [$y=-0.25x^2$]	
	Value of y [$y=0.25x^2-8$]		y is -8 (S) the coefficient at the equation [$y=0.25x^2-8$]	
	Distinguish top and bottom turning point [$y=7\sin(0.25\pi x)$]			
Horizontal stretch	Point where the graph changes orientation [$y=-0.25x^2$]			
Vertical reflection				Turning point changes [$y=0.25x^2-8$]

2 Constant function

Table 2.1
Summary of the perceptions of constant function in each microworld of the pairs of students who began by working with DG

	Bernard & Charles			Jane & Anne		
Function	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**	**FP**
y=6	y is motionless (D) 'It is useless to move x, the striker doesn't do anything' (D)	y is motionless (S) y is constant y is constant (S) (x,y) moves in horizontal straight line	Horizontal straight line (S) y is constant [DG]	y is motionless (D) x does not matter, the striker does not move from the same place (S) Constant function	y is always at 6 (D) (x,y) traces a straight line parallel to x-axis (S) constant function at 6.	A dot (A) Absence of x at the equation (A) x does not vary Horizontal straight line (S) y does not vary.
y=-3	y is motionless (D)	y is motionless(D)	Horizontal straight line (S) y is constant [DG]	The striker is motionless (D) y is negative while x is zero (G)	x varies but y is still -3 (D) Constant function at -3 (S) (x,y) traces a straight line parallel to x-axis.	Constant function (S) horizontal straight line Horizontal straight line (S) y does not vary.
y=2x						Variable (S) y varies in the graph [VS]

Table 2.2
Summary of the perceptions of constant function in each microworld of the pairs of students who began by working with FP

Function	John & Tanya			Diana & Gisele		
	FP	DG Parallel	DG Cartesian	FP	DG Parallel	DG Cartesian
y=6	Horiontal straight line (S) x can vary but y has just one value Absence of x at the equation (S) Horizontal straight line y has just one value (S) the function does not increase or decrease	y is motionless (S) y did not vary in a graph of straight line with null angle. Only x can change, y is motionless (D)		Constant function (S) y does not vary at graph (S) There is no x at equation	The striker does not move from the same pla ce (S) y does not vary (S) Constant function	y is constant (S) (x,y) moves in a horizontal line
y=-3		y is motionless (D). Only x can change, y is motionless (D)	Straight line with null angle [DG Par.] (S) x moves, moves but y does not move...	Only x varies and y is -3 (D) The function is neither increasing nor decreasing (D) y does not depend on x (S) There is no x at the equation [HT, HS and HR]	The striker does not move from the same place (S) it is constant	It is a constant function (S) x can go anywhere but y stays at -3

Diagram 2.1

Comparison of the students' perceptions of constant function

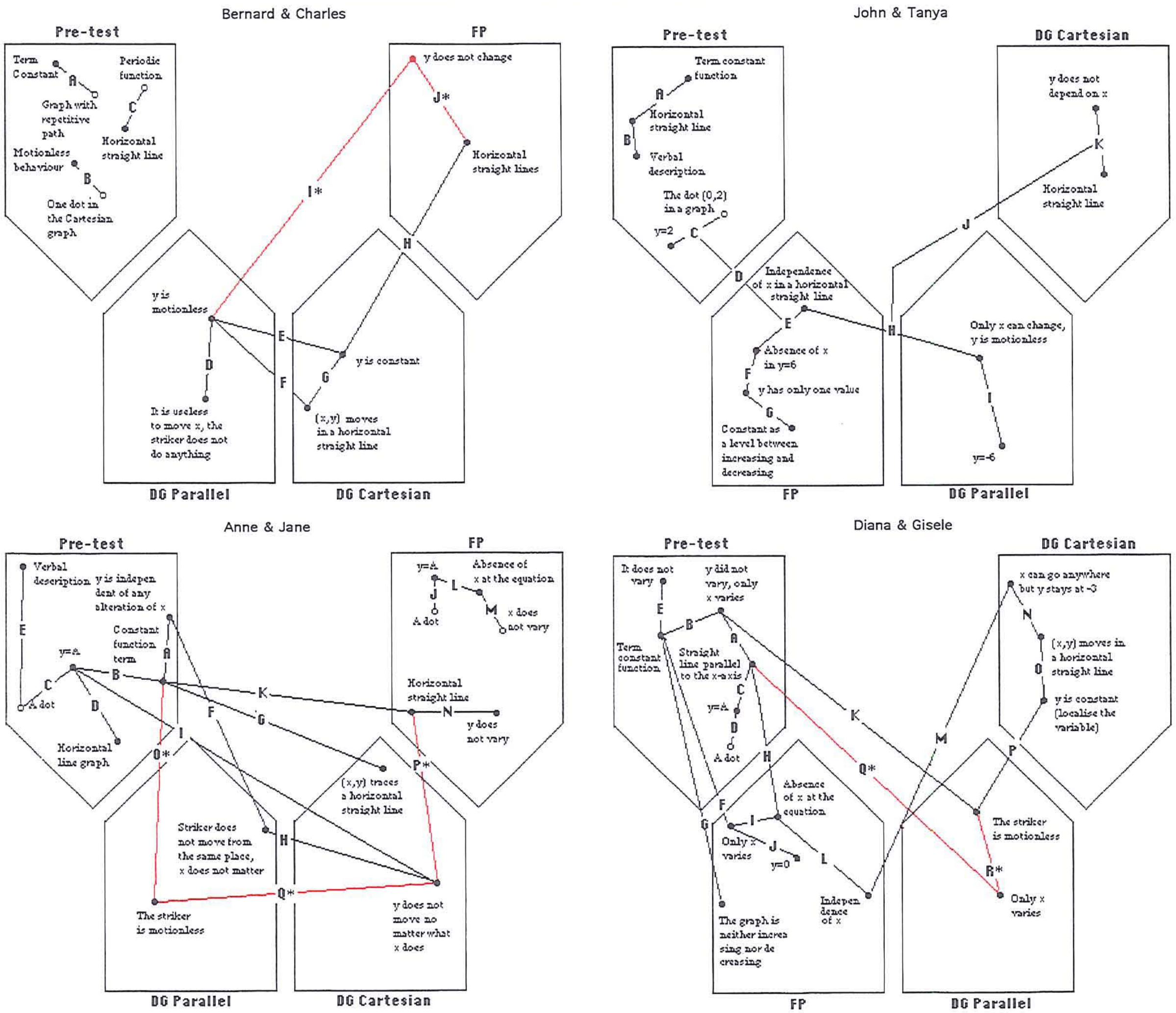


Table 2.3
Changes in students' perceptions of constant function in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Horizontal translation				Link 'the absence of x' in the equation to 'y is independent of x' [y=-3]
Horizontal stretch				Link 'the absence of x' in the equation to 'y is independent of x' [y=-3]
Vertical stretch		Localise the variable y as being the constant [y=2x]		
Horizontal reflection				Link 'the absence of x' in the equation to 'y is independent of x' [y=-3]

3 Monotonicity

Table 3.1

Summary of the perceptions of monotonicity in each microworld of the pairs of students who began by working with DG

Function	Bernard & Charles: DG Parallel	Bernard & Charles: DG Cartesian	Bernard & Charles: FP	Jane & Anne: DG Parallel	Jane & Anne: DG Cartesian	Jane & Anne: FP
y=x	The striker follows the triangle (x) (D)	The straight line grows to the right side (D)	Direction of straight line observing different slopes [HS] (G)	The striker follows the same orientation of x (G)	When x is going to positive, y is going to positive (D) Term increasing (S) x is increasing, y is increasing	Term increasing (S) direction of the straight line graph [HS]
y=-x	The striker moves in opposite orientation of the triangle (x) (D)	Direction of the straight line (A) the straight line is positive to the left side (D)	Direction of the straight line (D)	y follows different orientation of the triangle (D)	Term decreasing (S) Direction of the motion of (x,y)	Term decreasing (S) direction of the straight line graph [HS]
y=x-6		The term increasing (A) the straight line is positive after 6 (D)		The striker follows x in both sides (G)		Term increasing (S) direction of the straight line graph
y=2x	y and x move with the same orientation (D)		Direction of the straight line (D)	The striker follows only one orientation (G)	Term increasing (S) x is increasing, y is increasing	
$y=0.25x^2$	Sometimes the striker follows x, sometimes it does not (G)			At the positive side of x, 'y follows x'. (G)	y does not follow [always] x (D)	
$y=-0.25x^2$				At the positive side of x, 'y does not follow x' (G)		The graph stops growing and star ts decreasing (A) maximum
$y=0.5x^2$				At the positive side of the triangle, 'y follows x'. (G)		
$y=0.25x^2-8$				The striker has same orientation of x from the central point (A) when x>0, y>0		
y=7Sin(0.25πx)				When x>0, y can be positive or nega tive (A) when x>0, the orientation of y varies (D)		
y=7Sin(0.125πx)	The striker does not obey x (G)			When x is positive, y keeps changing the orientation (G)		

Table 3.2
Summary of the perceptions of monotonicity in each microworld of the pairs of students who began by working with FP

	John & Tanya			Diana & Gisele		
Function	**FP**	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**
y=x	Increasing (A) after x equal to zero, y is positive (D) Increasing (R) angle (S) Progressive [HT]	The striker obeys x (D)	Increasing (A) diagonal straight line from negative to positive of x and y (D)	A is positive (A) after the graph crosses x-axis, y is positive (A) direction (D)	The striker moves with the same orientation as the triangle (G)	Direction that (x,y) moves (S) term 'increasing' While x goes from positive to negative, y decreases, or vice-versa (G)
y=-x			Increasing in x and decreasing in y (D) Decreasing (A) y goes from positive to negative while x goes from negative to positive (D)	A is negative (A) after the graph intercepts x-axis, y is negative (D)	y always moves in opposite orientation to the triangle (G)	When the triangle increases, y decreases (D)
y=2x	Progressive in both axes (G)	y obeys the orientation of x (D)	y increases while x increases (G)	Direction of the straight line (A) A is positive (D)	y moves with the same orientation as the triangle [x] (G)	When x goes from negative to positive, (x,y) increases and vice-versa (D)
y=x-6	Progressive in both axes (G)	y obeys the orientation of x (D)	y increases while x increases (G)		y moves with the same orien tation as the triangle (G)	Direction of the straight line (D)
$y=0.25x^2$			Up to zero, it decreases in y and increases in x and after zero it increases in y and increases in x (G)	Increasing (A) A is bigger than zero (A) positive curvature (G)		When x is going from negative to positive (in the negative domain), y is decreasing (D)
$y=-0.25x^2$	Progressive up to x=0 to zero and regressive after x=0 [HS] (G)			Negative curvature (A) decreasing (D)	From negative up to zero the striker follows the triangle after zero it does not (D)	y is increasing without link to the variation of x (D)
$y=0.5x^2$		Sometimes y follows x, sometimes it does not - y follows x only in the positive domain (G)				Term increasing (A)positive curvature (D)
$y=0.25x^2-8$				The function is increasing (A) A is bigger than zero (A) positive curvature (D)	In the positive domain the striker follows the triangle up to -8, in the negative domain, the striker moves in opposite orientation (G)	Term increasing (A) positive curvature (D)
y=7sin (0.25πx)		Sometimes y follows x, sometimes it follows the opposite orientation of x (G)		The graph has positive and negative curvature (A) it is increasing and decreasing		
y=7Sin(0.125πx)	Changes many times between progressive and regressive (G)			The graph changes from cup to hill-shape (A) alternaly increases and decreases (G)	Sometimes the striker follows the triangle, sometimes it does not (G)	

Diagram 3.1

Comparison of the students' perceptions of monotonicity

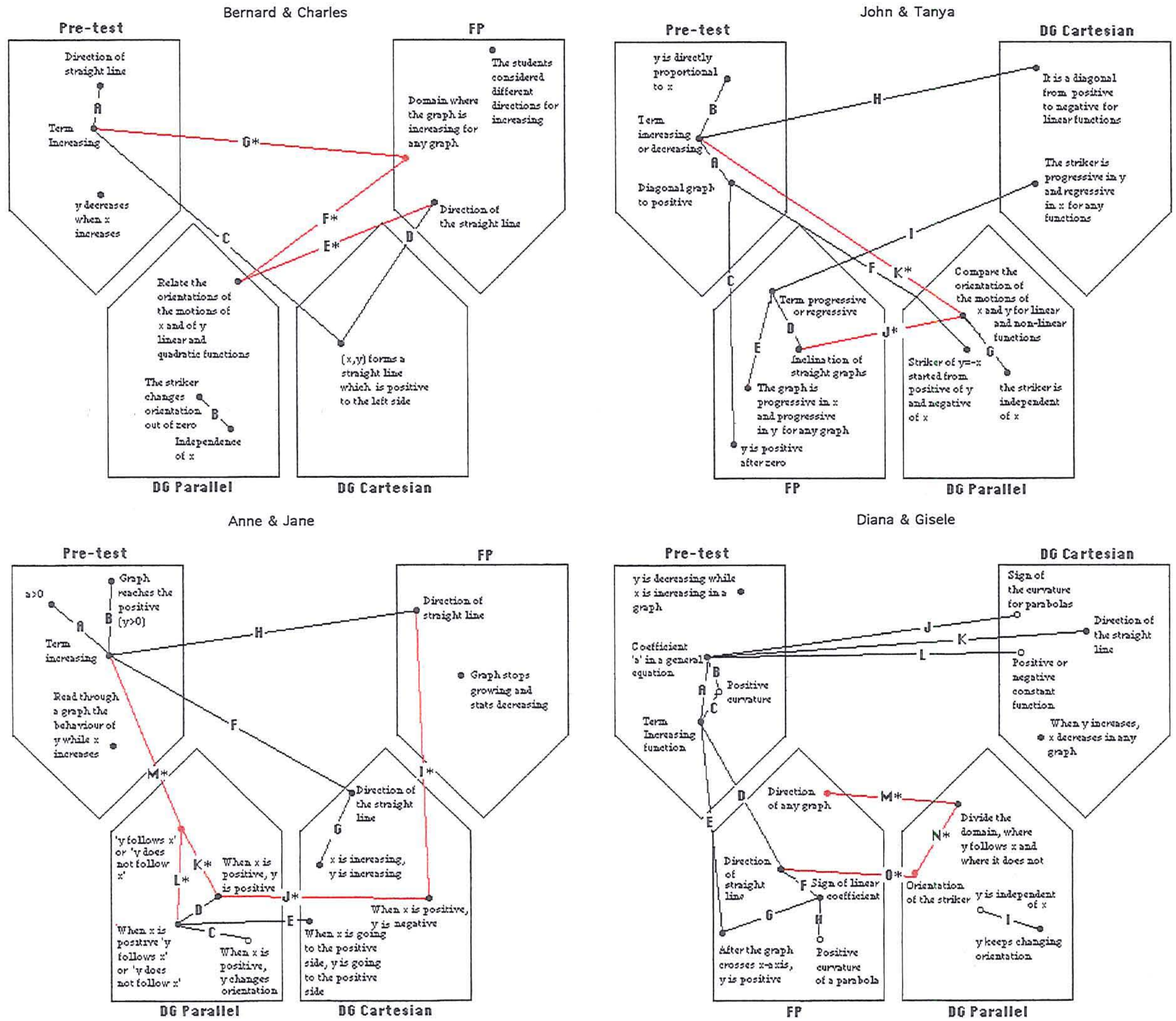


Table 3.3
Changes in students' perceptions of monotonicity in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Horizontal translation			Link 'progressive' in x and y to the inclination of straight line graphs [y=x]	
Horizontal stretch	Perceive the relationship between the ideas of slope and monotonicity in graphs of straight line [y=x]	Remember the term increasing by the inclination of the graph [y=-x and y=x]	Generalise increasing among parabolas calling it progressive [$y=-0.25x^2$]	

4 Derivative

Table 4.1

Summary of the perceptions of derivative in each microworld of the pairs of students who began by working with DG

	Bernard & Charles			Anne & Jane		
Function	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**	**FP**
y=x	The striker has the same speed as the triangle (x) (G)	The striker has the same speed as x (G) Proportion 1 to 1 (G)	Angular coefficient of the equation (S) Proportion 1 to 1 [HS][DG] Ratio between absolute values of y and x (S) Inclination [HS]	The striker is slower than the striker of y=2x (D) y has the same speed as x (D)	The striker has the same variation as x (S) y=x	Parallel straight lines (A) same equation (D) Parallel straight lines (S) same ratio between y-intercept and x-intercept [VT] abs(x)/abs(y) [HS](D)
y=-x	The striker moves with the same speed as the triangle (A) the striker and the tringle have the same distance to zero (D)	The striker has the same speed as the triangle (x) (D)		y has the same speed as x (G) (A) y and x have the same absolute value		Angle formed by the x-axis and the straight line [VS](D)
y=x-6	y has the same speed as x (D)	The striker has the same speed as x (G)	Proportion 1 to 1 (S) Parallelism with y=x Proportion 1 to 1 (S) Compare the strikers of y=x and y=x-6	y has the same speed as x (A) there is a gap between x and y (G)	The striker has the same variation as x (S) first degree polynomial function (S) Straight line	
y=2x	The striker is quicker than the triangle (D)	The striker is quicker than x (G) Proportion 2 to 1 (G)	Proportion of 2 to 1 (S) Coefficient of the equation [VT] 2 to 1 is the symmetric of the ratio between y-intercept and x-intercept [VT] (S)	The striker is quicker than the striker of y=x (D) y varies more than y of y=x (D) Abs(y) and Abs(x) (A) variation (A) How many y moves (G)		Angle formed by the straight line and x-axis (D)
$y=0.25x^2$	Striker is quicker than x (G)			y is slower than y of $y=0.5x^2$(D)	The striker runs away from the screen (D)	
$y=-0.25x^2$	Striker is quicker than x (D)					
$y=0.5x^2$	y is quicker than x (D)	y is quicker than x (D)		The striker is quicker than the striker of $y=0.25x^2$ (D)		
$y=0.25x^2-8$	Striker is quicker than the triangle (G)			The striker is quick (D)		
y=7Sin(0.25πx)		The striker is quicker than the striker of other sine(D)		The striker is quicker than y of y=2x (D) y is quicker than y of y=7sin(0.125πx) (D)	The variation of the striker is bigger than the variation of the other sine (D)	
y=7Sin(0.125πx)	The striker is quicker than the triangle (D)	The striker is slower than the striker of other sine(D)			The striker's variation is smaller than variation of the other sine (D)	

Table 4.2

Summary of the perceptions of derivative in each microworld of the pairs of students who began by working with FP

Function	John & Tanya FP	John & Tanya DG Parallel	John & Tanya DG Cartesian	Diana & Gisele FP	Diana & Gisele DG Parallel	Diana & Gisele DG Cartesian
$y=6$	Null Angle (D)					
$y=x$	Angle formed with the x-axis without considering the direction (monotonicity) (D) abs(x)/abs(y) is 1 to 1 (S) Inclination (angle) [VS]	Angle of 45 degrees (R) equation Angle of 45 degrees (S) y is over x y has the same speed as x (D)		The angle formed with x-axis is the same of $y=x-6$ (D)	The striker has the same speed as x (D)	The angles formed with the axes are equal (D)
$y=-x$	Angle formed with the x-axis without considering the direction (monotonicity) (D)	y has the same speed as x (D)			y runs almost the same as the triangle [x] (G)	
$y=2x$	Obtuse angle with the x-axis [VT] (S) y varies, it has many values	y is quicker than x (R) y can overtake x (D)	y grows one step, x grows half step (D) Angle with y-axis is smaller than 45 degrees (S) y is quicker than x.	Angle formed with the x-axis is bigger than the one formed with y-axis (D)[VS] Ratio between y-intercept and x-intercept is 2 (D) [VT] Ratio between the angles formed with the axes is 2 to 1 (A) 'a' is 2	y is quicker than the triangle [x] (D)	The angle formed with y-axis is smaller than the one formed with x-axis (A) y is always bigger than x (D)
$y=x-6$		Proportional (1 to 1) (S) when x moves 1 step; y moves 1 step	For each step x moves, y moves one step (D)	Angle with the x-axis is parallel to $y=x$ (G) (S) linear coefficient [VT] The absolute values of y-intercept and x-intercept are the same (G)	y has the same speed as the triangle (G)	
$y=-0.25x^2$						y runs more space than x does (D)
$y=0.5x^2$						y is quicker than y of $y=0.25x^2$ when it is going from zero to positive and slower when they appear again at the screen (A) 'Be in front of' (D)
$y=0.25x^2-8$			It is slower than $y=0.5x^2$ (D)·	y and x vary [VS to y=0] (D)	y is always quicker than x (D)	
$y=7\sin(0.25\pi x)$		y is quicker than y of the other sine (D)				The striker is quicker than the one of the other sine (D)
$y=7\mathrm{Sin}(0.125\pi x)$		y is very quick (D) y is quicker than other sine (D)			The striker is quicker than the triangle (D)	The striker is slower than the one of the other sine (D)

Diagram 4.1

Comparison of the students' perceptions of derivative

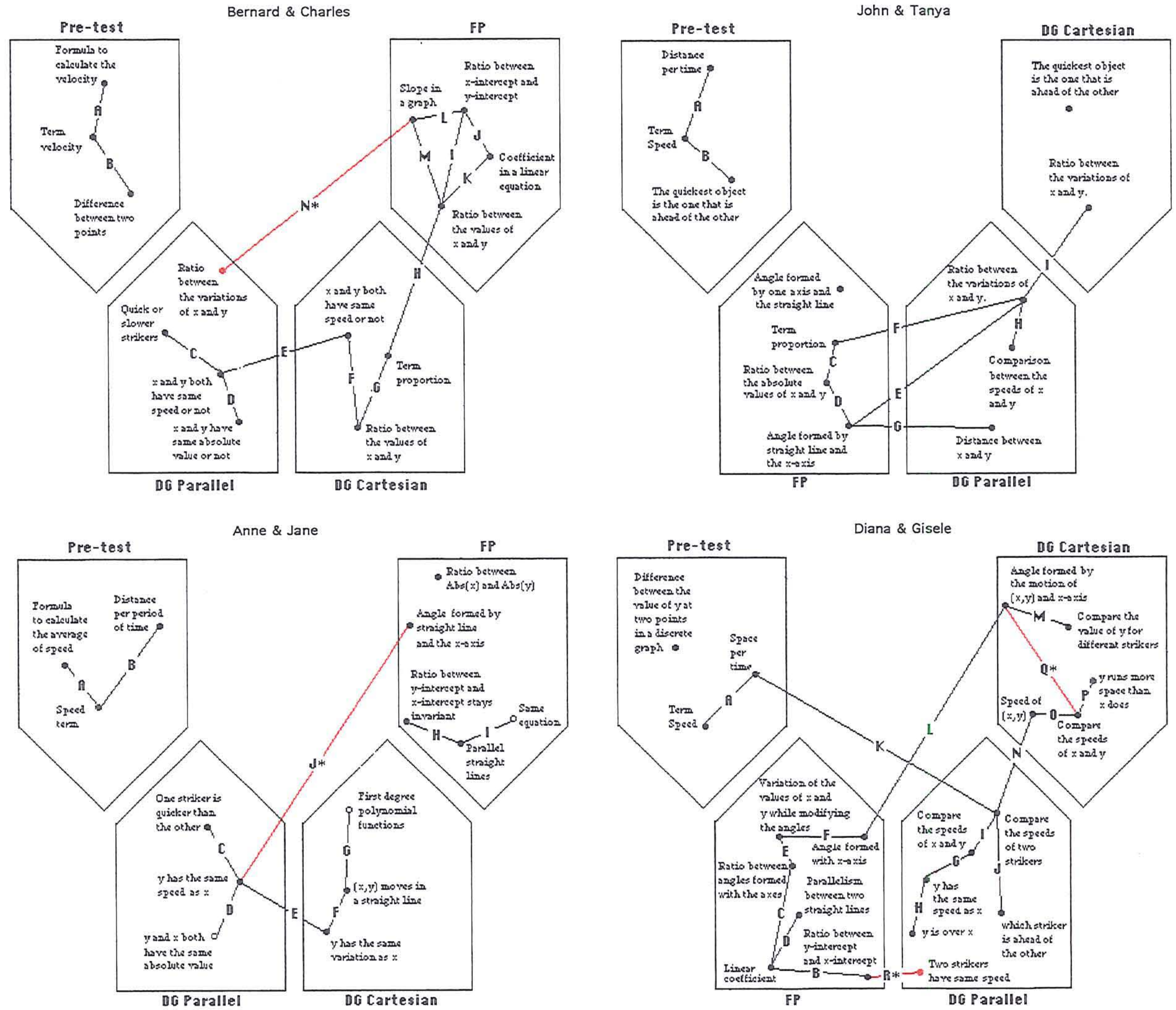


Table 4.3
Changes in students' perceptions of derivative in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Vertical translation	Realise that parallel straight lines have same ratio between x-intercept and y-intercept [y=2x]	Correspond 'parallelism between two straight lines' to same ratio between y-intercept and x-inter cept [y=x]	Generalise 'ratio between absolute values of x and y' to affine functions by linking it to the inclination of the graphs [y=x]	Generalise the link between linear coefficient and 'angle formed by the straight line and the x-axis' from 'linear' to affine functions [y=x-6]
	Link ratio between x-intercept and y-intercept, angular coefficient and ratio between the values of x and y [y=x to y=x-6]			Measure their idea of angle by ratio between y-intercept and x-intercept [y=2x]
Horizontal stretch	Link the linear coefficient to the ratio between the values of x and y [y=x to y=2x]	Perceive ratio between abs(x) and abs(y) as a invariant between [y=x and y=-x]		
	Realise that 'ratio between the values of x and y corresponds to the inclination of the graph [y=x]			
Vertical stretch		Observe the angle between the straight line and the x-axis [y=-x]	Perceive slope as order for monotonicity without measuring it [y=x]	Encourage the students to try a functional correspondence to the inclination of straight line graphs [y=2x]
			Build the perception of ratio between absolute values of x and y [y=x]	

5 Second derivative

Table 5.1

Summary of the perceptions of of second derivative in each microworld the pairs of students who began by working with DG

	Bernard & Charles			Jane & Anne		
Function	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**	**FP**
y=x	Constant speed (A) y and x have the same speed	Proportion1 to 1 (G)		The striker does not get speed (G)		
y=-x		The striker has the same speed as x (D)				
y=x-6		The striker has the same speed as x (G)				
y=2x	The striker slows down (A) it starts ahead of the one of $y=0.25x^2$ and this striker overtakes it	Proportion 2 to 1 (G)		The striker does not get speed (G)		
$y=0.25x^2$	The striker speeds up (A) it starts behind the striker of y=2x and it overtakes this striker	There is no fixed proportion between x and y (G) Term irregular proportion	Same curvature of the graph of $y=0.25x^2-8$ [VT] (D) Curvature (S) absence of proportion	The striker increases the variation (G)	The striker varies the varia tion less than in $y=0.5x^2$ (D) Graph bends less than the one of $y=0.5x^2$ (S) y goes slower	Graph has same openness as $y=-0.25x^2$ and $y=0.25x^2-8$ [VT](D) Proportionallity between two parabolas [VT]
$y=-0.25x^2$					(x,y) is moves bending (D)	Different curvatures (A) diffe rent distance between roots The graph has same openness as $y=0.25x^2$ (D)
$y=0.5x^2$			Same curvature of the graph of $y=0.5x^2-10$ [VT] (D) Compared with the curvature of the graph of $y=0.5(x/(-7))^2$ [HS] (D) Curvature (S) absence of proportion	The striker gets speed (increases the variation) (D)	The striker varies its variation more than the striker of $y=0.5x^2$ (D) The striker gets speed (S) (x,y) moves turning (x,y) traces a more bent graph than (x,y) of $y=0.25x^2$ (G)	The graph is more closed than $y=0.25x^2$ and $y=-0.25x^2$ (D)
$y=0.25x^2-8$	The striker stops near the turning point (D)	There is no fixed proportion (G)	Same curvature of the graph of $y=0.25x^2$ [VT] (D)			Same openness as $y=0.25x^2$ [VT] (D)
y=7sin (0.25πx)					(x,y) has curvature (S) y gets speed.	The graph is closed than y=7sin(0.125πx) (A) distance between the roots (D)
y=7Sin(0.125πx)						The graph is more open than y=7sin(0.25πx) (A) distance between the roots (D)

Table 5.2

Summary of the perceptions of second derivative in each microworld of the pairs of students who began by working with FP

Function	John & Tanya			Diana & Gisele		
	FP	DG Parallel	DG Cartesian	FP	DG Parallel	DG Cartesian
$y=2x$						y moves always one step (S)
$y=0.25x^2$	Same pattern of variation of x without same value of y (G) [VT from $y=0.25x^2-8$] [4]	The striker is slower than the one of $y=0.5x^2$ (S) Its graph is less bend than the graph of $y=0.5x^2$		Different curvatures (A) different angles [VS]	Up to 3, the striker is slower than x, it speeds up until it is quicker than x (A) y overtakes x (D)	y moves slower than x near x=0, later it moves quicker than x (D).
$y=-0.25x^2$	Same curvature [VT] [1] (D) 'x increases much faster than in the last one for the same y' [VS to $y= 0.0071(0.25x^2)$] (S) Different curvatures [2]			Same curvature (A) distance between symmetrical points for the same y (D)		
$y=0.5x^2$	Curvature is different from the one of the other parabolas (D)	The striker is quicker than the one of $y=0.25x^2$ (S) its graph is more bend than the graph of $y=0.25x^2$	The graph is more closed than the one formed by (x,y) of $y= 0.25x^2-8$ (A) distance between two symmetrical points			
$y=0.25x^2-8$	'Variation of x also increases equally' [VS to $y=-(0.25x^2-8)$ and VT to $y=-0.25x^2$] (G) Same curvature [3]	The striker becomes quicker each time (D)	The striker of $y= 0.5x^2$ is more closed than this one (S) distance between two symmetrical point is smaller then in the other striker	Curvature is more open than $y=(0.119)(0.25x^2 -8)$ with a proportionality between the difference of y for each x [VS] (S) for the same x, y is three times bigger than in the more open graph	y varies speed (D)	
$y=7\sin(0.25\pi x)$		y is quicker than $y=7\sin(0.125\pi x)$ (S) Its graph is more bend than the graph of $y=7\sin(0.125\pi x)$ (G)			y varies speed (D)	
$y=7\mathrm{Sin}(0.125\pi x)$		y is quicker than $y=7\sin(0.25\pi x)$ (S) Its graph is less bend than the graph of $y=7\sin(0.25\pi x)$ (G)		Bend of curvature (G)	y varies speed (D)	

Diagram 5.1

Comparison of the students' perceptions of second derivative

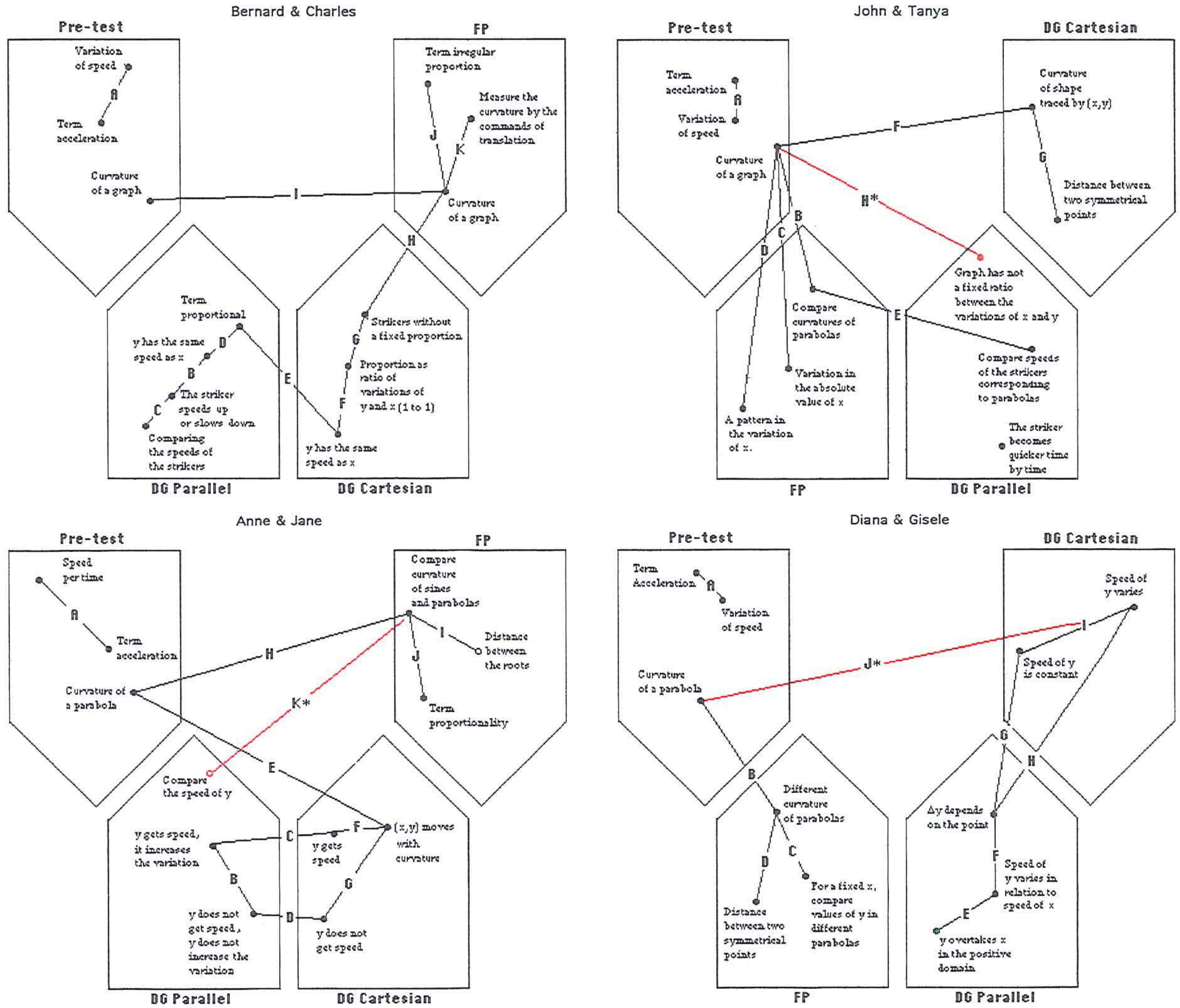


Table 5.3
Changes in students' perceptions of second derivative in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Vertical translation	Realise the invariance of the curvature [$y=0.25x^2$ and $y=0.5x^2$] Realise that the curvature cannot be measured by the distance between two symmetrical points [$y=0.25x^2$] Start to build a way to calculate curvatures which uses the command [$y=0.25x^2$]	Realise that two translated parabolas keep the same curvature [$y=0.25x^2-8$] Are encouraged to try a way to calculate the curvature of a parabola [$y= 0.25x^2-8$]	Realise the invariant curvatures [$y= -0.25x^2$] Realise that 'the variation of x increases' is invariant [$y=-0.25x^2$] Verify that there is a similar pattern in the variation of x [$y= 0.25x^2-8$]	
Horizontal stretch	Compare different curvatures of graphs [$y=0.5x^2$]			
Vertical stretch			Link the change in the curvature to 'x increases faster than the last graph for the same y' [$y=0.25x^2$] Realise that 'the variation of x' increases equally when reaching the symmetric graph [$y= 0.25x^2-8$]	Compare the curvature of parabolas [$y=0.25x^2-8$] Try a functional correspondence for parabolas with roots in the same points [$y=0.25x^2-8$] Realise that the curvature does not correspond to the 'length between two x for a fixed y [$y=7\sin(0.125\pi x)$] Associate curvature and angle [$y=0.25x^2$]

6 Range

Table 6.1
Summary of the perceptions of range in each microworld of the pairs of students who began by working with DG

	Bernard & Charles			Jane & Anne		
Function	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**	**FP**
y=6			y is positive [VT] (D)			y reaches only one value (D) There is no maximum or minimum (A) turning point
y=-3	The striker stays at the right side of the screen (D)		y is negative [VT] (D)			y reaches only one value (D)
y=x	The striker moves all the screen (G)	The striker moves all the graph (G)				
y=-x		The striker moves all y-axis (G)				y reaches many values (G)
y=x-6	The striker moves all the screen (G)	The striker moves all y-axis (G)				
y=2x	y moves all the screen (G)	Striker moves all y-axis (G)				
$y=0.25x^2$	The striker moves only in the right side (D)	The striker moves only to the middle of the screen (D)	y is only positive (A) Positive angular coefficient		(x,y) has a minimum (D)	Minimum (S) y of the turning point [VT and HT]
$y=-0.25x^2$	The striker moves only in the negative side (D) It moves only in one side (G)	y moves only in the negative numbers (D)		The striker retur ns at zero (D)	(x,y) has a maximum (D)	Maximum (R) y of the turning point
$y=0.5x^2$	The striker does not move in the negative side (D) It moves only in one side (G)	y moves only in the positive part of the y-axis (G)	y is only positive (A) Positive curvature			Minimum (R) y of the turning point
$y=0.25x^2-8$	The striker returns when it arrives at -8 (D) It does not go to the end of the axis (G) It does not go to one end of the screen (G)	The striker moves all y-axis (positive and negative) (G)				
y=7Sin(0.25πx)	Striker (y) returns when it arrives to some limit values (D)	The striker moves half of the axis (D)				The graph has same maximum and minimum as the graph of the other sine (A) turning point
y=7Sin(0.125πx)	The striker does not go to the end of the screen (both sides) (G)		Interval that its graph describes [VS] (D) Amplitude (R) height of turning points [HS] (D)			The graph has same maximum and minimum as the graph of the other sine (A) turning point Positive or negative (D) [VT]

Table 6.2

Summary of the perceptions of range in each microworld of the pairs of students who began by working with FP

Function	John & Tanya FP	John & Tanya DG Parallel	John & Tanya DG Cartesian	Diana & Gisele FP	Diana & Gisele DG Parallel	Diana & Gisele DG Cartesian
y=6	The value is 6 but its length is zero [VT] (G)	The value of y is 6 (D)	x is infinity while y does not move (G) y is in the positive side (D)	Positive value (A) A is positive (D)	The striker is positive (G)	y is 6 (D)
y=-3	The value is -3 but its length is zero [VT] (G)	The value of y is -3 (D)	y is in the negative side (D) x is infinity while y does not move (G)		The striker is negative (G) y is -3 (D)	y is -3 (D)
y=x	The range is infinity [VS] (G)				The striker runs all the line (G)	
y=-x					The striker runs all the line (G)	y is boundless (G)
y=x-6					y runs all the line (G)	
y=2x		y moves from negative infinity to positive infinity (R) y gets out of the screen before x (G)				
$y=0.25x^2$	y is only positive (A) Positive angular coefficient(D)			Graph passes in the first quadrant (D)	y is only positive (D)	y runs only in the positive side (D)
$y=-0.25x^2$	Length of the set composed by y [VR] (G) y is only negative (D)		y goes from negative infinity to zero (G)	The graph is negative (A) A is negative (D)		y does not pass to positive side (D)
$y=0.5x^2$	Angular coefficient is negative (A) y is just negative (D)	y goes from zero to positive infinity (G)		The minimum of the parabola is at the turning point (D)	y runs only in the positive side (D)	Parabolas have minimum (R) the turning point (D)
$y=0.25x^2-8$	Length of the set composed by y is infinity [VR] (D) y that the graph can reach (G)	y is bigger than -8 (D) Bounded (A) Motion of y striker (D)			The striker moves in positive and negative (G) y comes up to a point and returns (G)	Parabolas have minimum (A) the turning point is negative (D)
$y=7Sin(0.25\pi x)$	Place where y can be given by the turning points [VT] (D) Two aspects: 'extension and position' of range [VS] (D)		y moves from -7 to 7 (D)	The quadrants it passes (A) It is positive and negative (D)	y moves between -7 an 7 (D)	
$y=7Sin(0.125\pi x)$	Same extension and position of the other sine [VS] (D)	y stays between -8 and 8 (G) y does not moves out of -8 and 8 (D)	y moves from -7 to 7 (G)	It has maximum and minimum (D) The height of the graph (D)	y moves between -7 and 7 (D)	

Diagram 6.1
Comparison of the students' perceptions of range

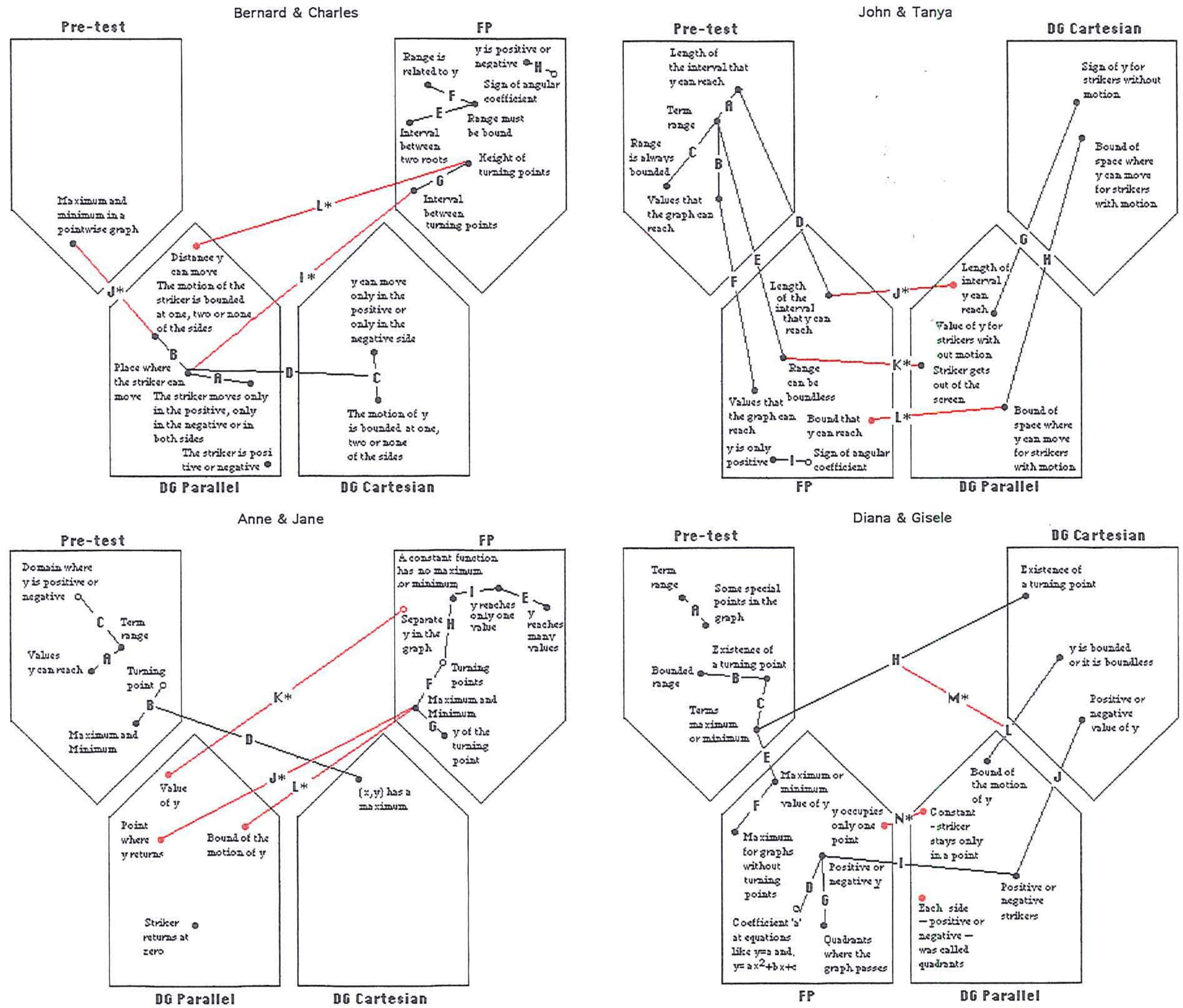


Table 6.3
Changes in students' perceptions of range in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Horizontal translation		Distinguish the ideas of maximum as y of the turning point from the turning point [$y=0.25x^2$]		
Vertical translation	Separate the range into positive and negative [y=6 and =-3] Revise the association between positive or negative range and sign of angular coefficient [$y=0.25x^2$-8 to $y=0.25x^2$]	Link maximum to y of the turning point [$y=0.25x^2$] Explore the range as positive or negative [$y=7\sin(0.125\pi x)$]	Distinguish amplitude of range from the range [$y=7\sin(0.25\pi x)$] Discuss the infinitude of range on Cartesian graph [$y=0.25x^2-8$] Revise the association between positive angular coefficient and positive range [$y=0.25x^2$]	Distinguish range from amplitude of range in graph by linking with interval and length of interval in DG parallel [$y=7\sin(0.25\pi x)$] [FI]
Horizontal stretch	Discriminate two ideas related to range: amplitude and range [$y= 7\sin(0.125\pi x)$]			
Vertical stretch	Discriminated range as the interval between turning points [$y= 7\sin(0.125\pi x)$]		Distinguish amplitude of range from range [$y=7\sin(0.25\pi x)$] Realise amplitude of range of linear functions (non-constants) as being infinity [y=x]	Link amplitude of range in a graph with length of interval the striker moves [$y=7\sin(0.25\pi x)$] [FI]
Vertical reflection			Considered the length of the set that y can reach [$0.25x^2-8$]	

7 Symmetry

Table 7.1
Summary of the perceptions of symmetry in each microworld of the pairs of students who began by working with DG

	Bernard & Charles			Jane & Anne		
Function	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**	**FP**
y=x			Symmetric to y=-x (S) $f_1(-x)=f_2(x)$ [VS]			It is symmetric to y=-x (A) Symmetric numbers.
y=-x	y is symmetric to x (D)		Symmetric to y=x (S) $f_1(-x)=f_2(x)$ [VS]			y=-x is symmetric (R) symmetric numbers.
y=x-6			The graph crosses the x-axis and y-axis in symmetric values (D)			
$y=-0.25x^2$			Line symmetry in the y-axis (G)			
$y=0.5x^2$			Line symmetric graph (A) $f(x)$ must correspond to $f(-x)$ Line of symmetry is in the y-axis [HT] (G)			

Table 7.2
Summary of the perceptions of symmetry in each microworld of the pairs of students who began by working with FP

	John & Tanya			Diana & Gisele		
Function	**FP**	**DG Parallel**	**DG Cartesian**	**FP**	**DG Parallel**	**DG Cartesian**
y=6	Symmetric to y=0 with line of symmetry at y=3 [VT] (D)					
y=-x	Positive and negative numbers [VR] (D)	x and y are opposite (D)	y and x stay on symmetric numbers (D)			
$y=-0.25x^2$	Line symmetry (S) $f(x)=f(-x)$ Line symmetry (R) the graph has two equal sides [HR]			Line symmetry out of the y-axis (D) [HR]		
$y=0.25x^2-8$				Line symmetry out of the x-axis (S) y of the turning points are not symmetric numbers [VR]		
y=7Sin(0.25πx)		Striker alternates between 'a number and its symmetric' (D)		Line symmetry at any turning point (D)		The minimum keeps repeating (D)
y=7Sin(0.125πx)				Line symmetry passing through any of its turning points (D)		Line of symmetry in the turning point (D)

Diagram 7.1

Comparison of the students' perceptions of symmetry

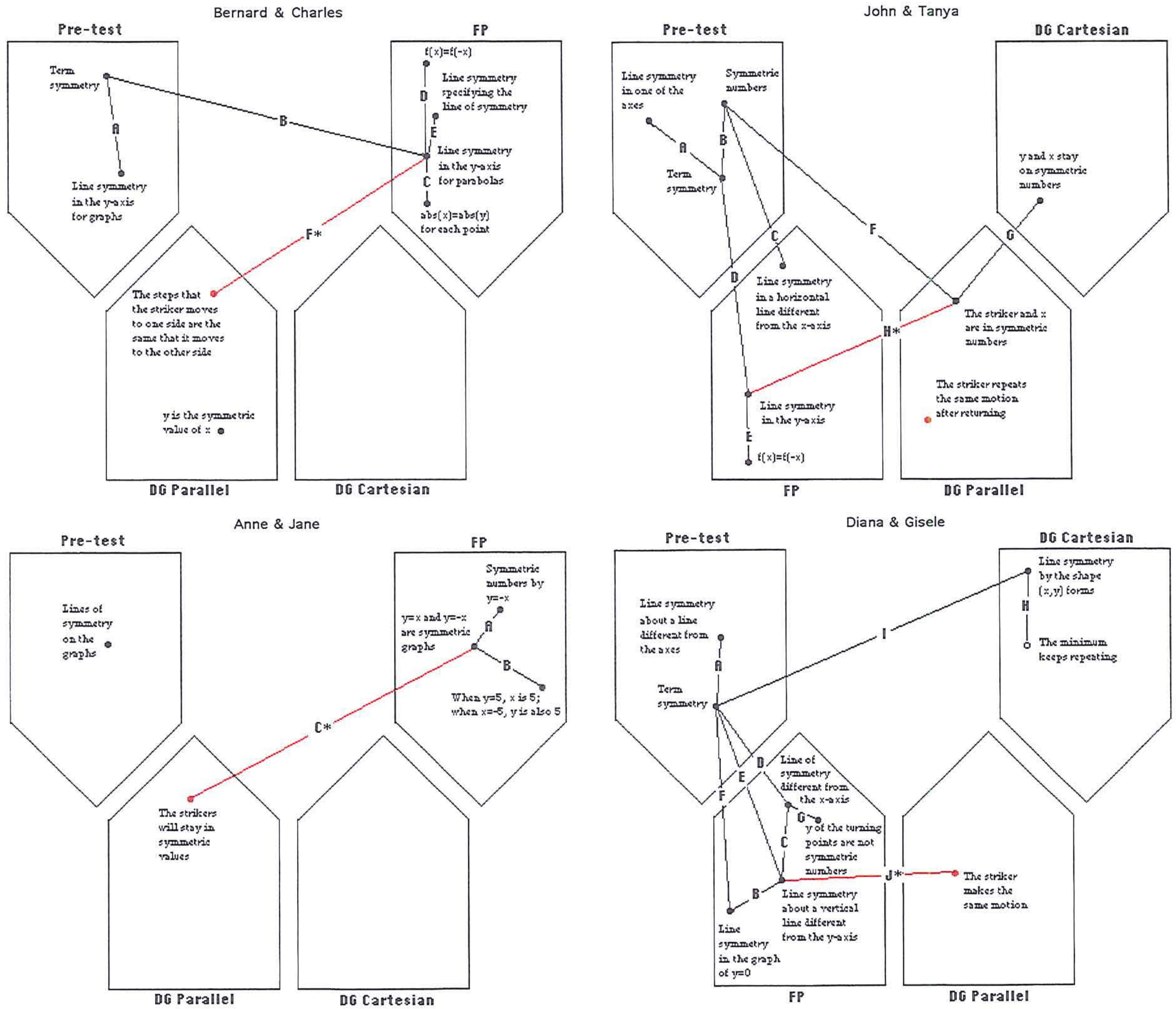


Table 7.3

Changes in students' perceptions of symmetry in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Horizontal translation	Start to identify line of symmetry [y=0.5x^2]			
Vertical translation	Search for a functional correspondence of line symmetry [y=abs(x)]		Identify line symmetry out of the x-axis [from y=0 to y=6]	
			Revise the association between line symmetry and symmetric numbers	
Vertical stretch	Link line symmetry on the y-axis to $f_1(-x)=f_2(x)$ [y=x and y=-x]			Discuss the symmetry of the graph y=0 [y=abs(x) to y=0]
Horizontal reflection			Justify the invariance of the graph by the line symme try [y=-0.25x^2]	Identify line symmetry out of the y-axis [y=0.25x^2 to y=0.25$(x+14)^2$]
Vertical reflection			Identify symmetric numbers [y=-x]	Identify line symmetry out of the x-axis and conclude that y of turning points do not need to be symmetric numbers [y=0.25x^2-8 to y=-0.25x^2+11.6]

8 Periodicity

Table 8.1

Summary of the perceptions of periodicity in each microworld of the pairs of students who began by working with DG

Function	Bernard & Charles: DG Parallel	Bernard & Charles: DG Cartesian	Bernard & Charles: FP	Jane & Anne: DG Parallel	Jane & Anne: DG Cartesian	Jane & Anne: FP
y=7Sin(0.25πx)		The frequency of turning points is bigger than in the other striker of sine (D)	Distance between two roots [HT] — It repeats from 8 to 8 units (D) Invariant period calculated in turning points of -7 - It is from 8 to 8 [HT] (G)	y oscillates (A) y is independent of x (D)	Repetitive roots and the sign after each root (S) Oscillatory	Distance between two roots stays the same (D)
y=7Sin(0.125πx)		The frequency of turning points is smaller than in the other striker of sine (D)	The turning points of -7 repeats from 16 to 16 (S) the term period	y continues changing orientation at 6 and -6 (D)	Many parabolas (D) Minima and Maxima are equal but at different times (D)	Distance between two roots stays the same (D)

Table 8.2

Summary of the perceptions of periodicity in each microworld of the pairs of students who began by working with FP

Function	John & Tanya: FP	John & Tanya: DG Parallel	John & Tanya: DG Cartesian	Diana & Gisele: FP	Diana & Gisele: DG Parallel	Diana & Gisele: DG Cartesian
y=7Sin(0.25πx)	Period (R) Revolution of the graph in each domain (A) Interval each revolution takes (D) Period (R) Each revolution repeats after 8 units [VS] (D)	Path of y repeats after 8 steps x moves (D)	Period is smaller than period of the other sine (D)	The trace repeats af ter some length (A) the roots repeat (D)	The oscillatory behaviour of the striker (D)	y always repeats the same interval (D) The point makes a turn each 4 units that x moves (D)
y=7Sin(0.125πx)	Period (R) Each revolution repeats after 16 units (S) Coefficient that multiplies x [VS]	Path of y repeats after 16 steps x moves (D)	Periodic function (R) (x,y) repeats its revolution (D) Period is twice the period of the other sine (D)	The trace repeats after some time (D)	Oscillation of 'y follows x' and 'y does not follow x' (S) 'y is independent of x'	y always repeats the same interval (D) The point makes a turn each 8 units that x moves(G)

Diagram 8.1

Comparison of the students' perceptions of periodicity

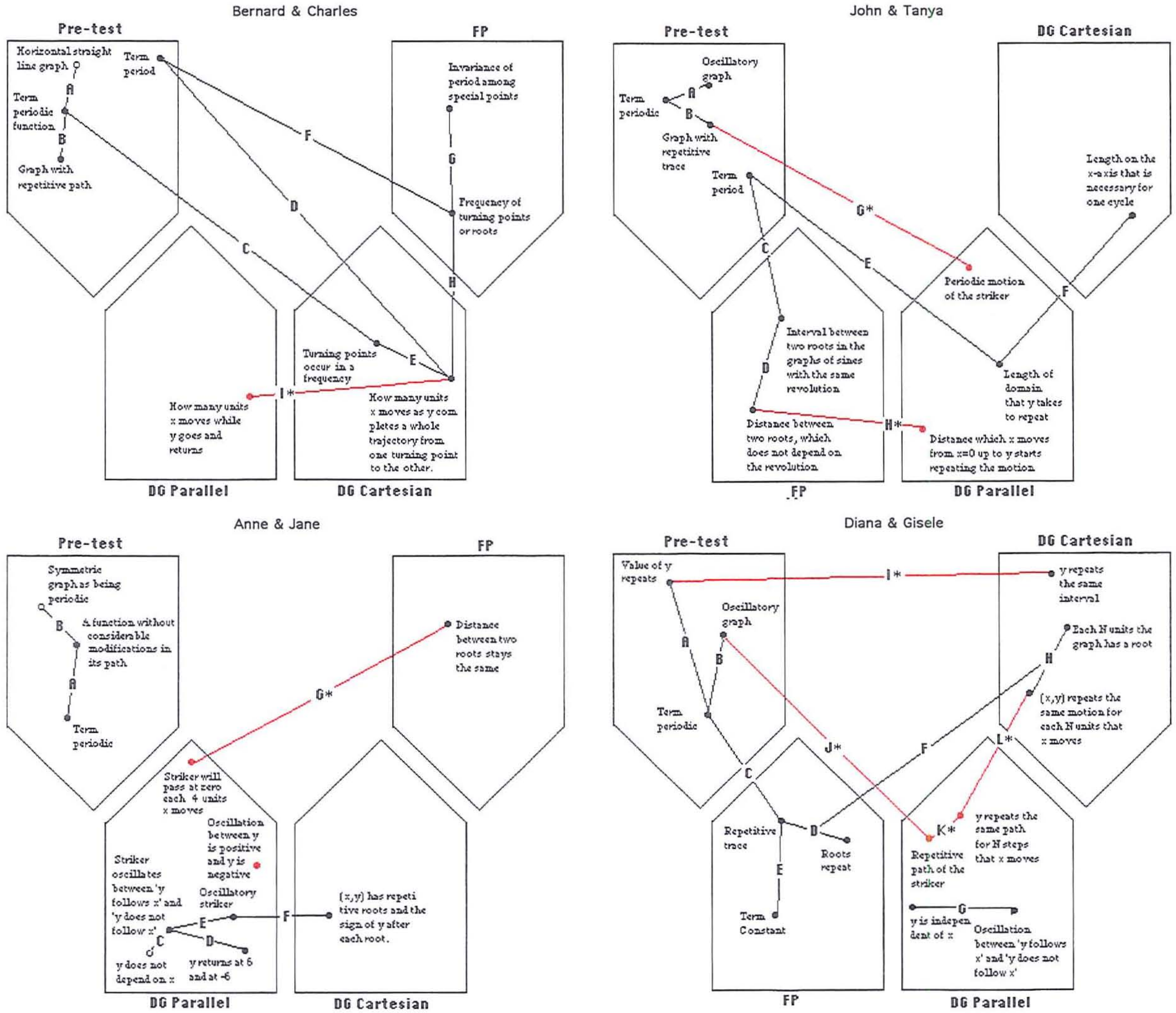


Table 8.3

Changes in students' perceptions of periodicity in relation to FP commands

Commands	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Horizontal translation			Progress from the perception of period as the interval between two roots to perceive it as the distance between two roots [$y=7\sin(0.25\pi x)$]	
Vertical translation	Recognise the invariance of the period calculated in special points [$y=7\sin(0.25\pi x)$]			
Vertical stretch		Link '(x,y) repeats same path each N units that x moves' in graphs to 'y repeats the same path for N steps that x moves' in DG Parallel in the final interview [$y=7\sin(0.25\pi x)$]	Recognise period as being how many units x moves for one revolution and revising the association betwee period and cycles [$y=7\sin(0.25\pi x)$] Link period with coefficient that multiplies x at the equation [$y=7\sin(0.25\pi x)$]	

Appendix V - Tables Summarising the Students' Findings in Different Properties

1 Findings while exploring transformations of graphs

Table 1.1
The students' findings while exploring the horizontal translation

Commands / Properties	Horizontal translation: Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Turning point			The point where a graph changes from increasing to decreasing [y=abs(x)]	Reveal to me the association between turning point and extreme values - It does not change [$y=0.25x^2-8$]
Constant function				Link 'the absence of x' in the equation to 'y is independent of x' [y=-3]
Monotonicity			Link 'progressive in x and y to the inclination of straight line graphs [y=x]	
Range		Distinguish the ideas of maximum as y of the turning point from the turning point [$y=0.25x^2$]		
Symmetry	Start identifying line of symmetry [$y=0.5x^2$]			
Periodicity			Progress from the perception of period as the interval between two roots to perceive it as the distance between two roots [$y=7\sin(0.25\pi x)$]	

Table 1.2
The students' findings while exploring the vertical translation

Commands		Vertical translation		
Properties	Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Turning point	Coordinate [$y=0.5x^2$] Value of y [$y=0.25x^2-8$] Distinguish top and bottom turning point [$y=7\sin(0.25\pi x)$]		Top of parabola [$y=-0.25x^2$] The value of y is -8 (S) the coefficient at the equation [$y=0.25x^2-8$]	
Derivative	Realise that parallel straight lines have same ratio between x-intercept and y-intercept [y=2x] Link ratio between x-intercept and y-intercept, angular coefficient and ratio between the values of x and y [y=x to y=x-6]	Correspond 'parallelism between two straight lines' to same ratio between y-intercept and x-intercept [y=x]	Generalise 'ratio between absolute values of x and y' to affine functions by linking it to the inclination of the graphs [y=x]	Generalise the link between linear coefficient and 'angle formed by the straight line and x-axis' from 'linear' to affine functions [y=x-6] Measure their idea of angle by ratio between y-intercept and x-intercept [y=2x]
Second derivative	Realise the invariance of the curvature [$y=0.25x^2$ and $y=0.5x^2$] Realise that the curvature cannot be measured by the distance between two symmetrical points[$y=0.25x^2$] Start to build a way to measure curvatures using the command [$y=0.25x^2$]	Realise that two translated parabolas have the same curvature [$y=0.25x^2-8$] Try a way to measure to the curvature of a parabola [$y= 0.25x^2-8$]	Realise the invariant curvatures [$y=-0.25x^2$] Realise that 'the variation of the increase of x' is invariant [$y=-0.25x^2$] Verify that there is a similar pattern in the variation of x [$y= 0.25x^2-8$]	
Range	Separate the range into positive and negative [y=6 and =-3] Revise the association between positive or negative range and sign of angular coefficient [$y=0.25x^2-8$ to $y=0.25x^2$]	Link maximum to y of the turning point [$y=0.25x^2$] Explore the range as positive or negative [$y= 7\sin(0.125\pi x)$]	Distingush the amplitude of the range from the range [$y=7\sin(0.25\pi x)$] Discuss the infinitude of range on Cartesian graph [$y=0.25x^2-8$] Revise the association between positive angular coefficient and positive range [$y=0.25x^2$]	Distinguish range from amplitude of range in the graph by linking with interval and length of interval in DG Parallel [$y=7\sin(0.25\pi x)$] [FI]
Symmetry	Try a functional perception of the line symmetry [y=abs(x)]		Identify line symmetry out of the x-axis [from y=0 to y=6] and overcome the association between line symmetry and symmetric numbers	
Periodicity	Recognise the invariance of the period calculated in special points [$y=7\sin(0.25\pi x)$]			

Table 1.3
The students' findings while exploring the horizontal stretch

Command / Properties	Horizontal stretch: Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Turning point	Point where the graph changes orientation [$y=-0.25x^2$]			
Constant function				Link 'the absence of x' in the equation to 'y is independent of x' [y=-3]
Monotonicity	Perceive the relationship between the ideas of slope and monotonicity in graphs of straight line [y=x]	Remember the term increasing by the inclination of the graph [y=-x and y=x]	Generalise the idea of progressive to parabolas [$y=-0.25x^2$]	
Derivative	Link the linear coefficient to the ratio between the values of x and y [y=x to y=2x] Realise that 'ratio between the values of x and y corresponds to the inclination of the graph [y=x]	Perceive ratio between abs(x) and abs(y) as an invariant between [y=x and y=-x]		
Second derivative	Compare different curvatures of the graphs [$y=0.5x^2$]			
Range	Distinguish two ideas related to range: amplitude and range [y= 7sin(0.125πx)]			

Table 1.4

The students' findings while exploring the vertical stretch

Commands	Vertical stretch			
Properties	**Bernard & Charles**	**Anne & Jane**	**John & Tanya**	**Diana & Gisele**
Constant function		Localise the variable y as being the constant [y=2x]		
Derivative		Observe the angle between the straight line and x-axis [y=-x]	Perceive slope as order for increasing and decreasing without measuring it [y=x] Build the perception of ratio between absolute values of x and y [y=x]	Try a functional meaning to the inclination of straight line graphs [y=2x]
Second Derivative			Link the change in the curvature to 'x increases faster than the last graph for the same y [$y=0.25x^2$] Realise that the variation of x increases equally when reaching the symmetric graph [$y= 0.25x^2-8$]	Compare the curvatures of parabolas [$y=0.25x^2-8$] Try a functional view for parabolas with roots in the same points [$y=0.25x^2-8$] Realise that the curvature does not correspond to the 'the distance between two symmetrical points' [$y=7\sin(0.125\pi x)$] Associate curvature and angle [$y=0.25x^2$]
Range	Discriminate the range as the interval between turning points [$y=7\sin(0.125\pi x)$]		Distinguish the amplitude of the range from the range [$y=7\sin(0.25\pi x)$] Realise the amplitude of range of linear functions as being infinity [y=x]	Link amplitude of range in a graph with length of interval the striker moves [$y=7\sin(0.25\pi x)$] [FI]
Symmetry	Link line symmetry in the y-axis to $f_1(-x)=f_2(x)$ [y=x and y=-x]			Discuss the symmetry of the graph y=0 [y=abs(x) to y=C]
Periodicity		Link '(x,y) repeats same path each N units that x moves' in graphs to 'y repeats the same path for N steps that x moves' in DG Parallel [$y=7\sin(0.25\pi x)$] [FI]	Recognise period as being 'how many units x moves for one cycle' and revise the period as being 'the cycle' [$y=7\sin(0.25\pi x)$] Link the period with the coefficient that multiplies x at the equation [$y=7\sin(0.25\pi x)$]	

Table 1.5
The students' findings while exploring the horizontal reflection

Command / Properties	Horizontal reflection: Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Constant function				Link 'the absence of x' in the equation to 'y is independent of x' [y=-3]
Symmetry			Justify the invariance of the graph by the line symmetry [$y=-0.25x^2$]	Identify line of symmetry different from the y-axis [$y=0.25x^2$ to $y=0.25(x+14)^2$]

Table 1.6
The students' findings while exploring the vertical reflection

Command / Properties	Vertical reflection: Bernard & Charles	Anne & Jane	John & Tanya	Diana & Gisele
Turning point				Turning point changes [$y=0.25x^2-8$]
Range			Consider the length of the set that y can reach [$0.25x^2-8$]	
Symmetry			Identify symmetric numbers [y=-x]	Identify line symmetry out of the x-axis and conclude that y of turning points do not need to be symmetric numbers [$y=0.25x^2-8$ to $y=-0.25x^2+11.6$]

Codes used in table 1.7

(1) The students discovered new perceptions or important aspect of a known property while transforming graphs or while understanding results of a transformation
(2) The command was used in a way that revealed to the researchers associations students brought - Lenses
(3) The students created critical moments to revise associations between two different properties and distinguished them
(4) The students linked the invariants or variants in the continuous modification of the graph
(5) The students linked the algebraic and graphic representations while exploring the command
(6) The students generalised one perception for different functions or for a different way of perceiving them
(7) The students used the commands to develop a measure for previous perceptions
(8) The students were led into a functional search of the pictorial perceptions
(9) The students used the transformations strengthening one association or creating a new one
(10) The students were only reporting aspects which were variant or invariant under the transformations
(11) The students overcame the compartmentalisation of knowledge perceiving two properties as a connected set of ideas
(12) The students started to separate the variables x, y and (x,y)

Table 1.7
The main ways the pairs used the transformations

Properties	Pairs	Vertical translation	Horizontal translation	Vertical stretch	Horizontal stretch	Vertical reflection	Horizontal reflection
Turning point	B&C	1			10		
	J&A						
	J&T	4 5	1				
	D&G		2			10	
Constant Function	B&C						
	J&A			1 12			
	J&T						
	D&G		4 5				1 5
Monotonicity	B&C				11		
	J&A				10		
	J&T		4 8		6		
	D&G						
Derivative	B&C	4 5 7 9			4 5		
	J&A	4 7 9		10	10		
	J&T	7 9		8 11			
	D&G	4 5 7 9		8			
Second Derivative	B&C	2 3 4 7			10		
	J&A	4 7 8					
	J&T	4 7 8		4 8			
	D&G			1 2 3 8 9			
Range	B&C	9		3	1		
	J&A	2 9 12	2 3 12				
	J&T	1 4		1 3 6		1	1
	D&G	4					1
Symmetry	B&C	8	1	8			
	J&A						
	J&T	3 6				10	
	D&G			1 6		1	
Periodicity	B&C	6					
	J&A						
	J&T		3	1 2 3 5			
	D&G						

2 Synthesis

Codes used in table 2.1:

PK - Link with previous knowledge

FP - Link with perceptions derived from interaction with Function Probe

Table 2.1

Connections built while working in DG Parallel in each property

Property	Students in DG to FP	Students in FP to DG
Turn. point		
Constant function	[PK] Recognised constant function by 'y does not move no matter what x does' [Terms] [J&A]	[FP] Linked 'y did not vary in a horizontal straight line' to 'y is motionless [J&T] [PK] Recognised constant function through 'y does not vary' to 'the striker does not get out of the same place' [D&G]
Monotonicity		[PK] Linked 'diagonal graph from negative' to 'the striker started from positive of y and negative of x' [J&T]
Derivative		[FP] Linked 'angle formed between straight line and the x-axis' to 'ratio between the variations of x and y' [J&T] [FP] Linked temporarily 'angle formed between straight line and the x-axis' and 'imaginary angle formed in DG Parallel' [J&T] [NL] [FP] Brought the term 'proportion' to indicate 'when the striker moves one step, y moves one step' [Terms] [J&T] [PK] Used the definition 'space per time' to decide about the speed of the strikers [D&G]
Sec.Deriv.		[FP] Linked different curvatures to different speeds of the strikers of parabolas [J&T]
Range	[PK] Used the idea of infinity to explain the behaviour of the striker when it leaves the screen [B&C]	[PK] Used infinity to explain the behaviour of the striker when it leaves the screen [J&T] [FP] Positive or negative y was compared with positive or negative striker [Pointwise] [D&G]
Symmetry		[PK] Used symmetric numbers to characterise x and the striker in y=-x [J&T]
Periodicity		[PK] Brought the term period to make sense in DG Parallel [J&T]

Codes used in table 2.2

(A) Match striker and family of functions

(B) Match strikers from DG Cartesian to strikers of DG Parallel

(C) Possibility of observing x, y and (x,y) separately

(D) Bring mathematical terms to make sense in DG Cartesian

(E) Match strikers and graphs of FP

(F) Contrast presence and absence of shape

(G) Examine counter-examples

(H) Compare or classify strikers

(I) Sketch graph of the striker

Table 2.2

Connections built while working in DG Cartesian in each property

Properties	Connections with DG Parallel	Connected both	Connection with graphs in previous knowledge or FP	Bridges DG -> Cart.	Bridges Cart. -> DG	Motivation
	Point where strikers change orientation [D&G]	[D&G]	Shape traced by (x,y) [D&G] [J&A] [B&C]	[D&G]		A
Turning points	Point where strikers change orientation [J&T]	[J&T]	The point where the graph changes from increasing to decreasing [J&T]	[J&T]		A
	Bound of the motion of y [J&T]					B
	The striker is motionless [B&C] [D&G]	[B&C] [D&G]	(x,y) moves in horizontal straight line (S) y is constant [B&C] [D&G]	[B&C] [D&G]		A C
Constant functions	y does not depend on x [J&T]	[J&T]	(x,y) moves in a horizontal straight line [J&T]	[J&T]		A C
	The striker does not depend on x [J&A] [D&G]	[D&G] [J&A]	y=a [J&A]; y is independent of x and the term constant function [D&G] [J&A]		[D&G]	A C D A
	y is motionless (S) (x,y) is motionless (Overcame) (NL) [B&C] [D&G]		(x,y) moves in a horizontal straight line [J&A]			B
			(x,y) forms a straight line which is positive to left side (S) Term increasing [J&T] [B&C]			A D
			Direction of straight line formed by (x,y) (s) sign of linear coefficient [D&G]			A D
Monotonicity			The striker is progressive in y and regressive in x [J&T]			E
	'y follows x' (S) when x is going to the positive side, y is going to the negative side [J&A]					B
			Term increasing (S) (x,y) forms a diagonal straight line and x is increasing, y is decreasing [J&A]		[J&A]	F
	'Ratio between absolute values of x and y' [D&G] [B&C]					B
	'Ratio between variations of x and y' [J&T]					B
Derivative	Changing from 'Ratio between absolute values..' to '... between variations ...' [B&C] while working in FP	[B&C] in FP	Slope of graph formed by (x,y) [B&C]	[B&C] in FP		G
			Inclination of graph (S) comparison between x and y [D&G]		[D&G]	F
	'y and x has the same speed' [J&A]	[J&A]	Inclination of graph traced by (x,y) [J&A]	[J&A]		C
	'y gets speed' or 'y does not get speed' [J&A]	[J&A] in FP	(x,y) moves turning (S) Curvature of a graph [J&A]	[J&A] in FP	[J&A] in FP	A E
Second Derivative	Fixed or variable ratio between variations of x and y [B&C]	[B&C] in FP	Curve or straight line [B&C] in FP	[B&C] in FP		E F
	Fixed or variable ratio between variations of x and y [D&G]					B
			Curvature of shape traced by (x,y) (S) distance between two symmetrical points [J&T]		[J&T]	G
			(x,y) has extreme values (S) turning point [J&A] [D&G]			A D
	Positive or negative for motionless strikers [J&T]					B
Range	Bounded or boundless place where striker can move [J&T]					C H
	Polarised division for place where striker can move [B&C] [D&G]					C
	Bounded or boundless place where striker can move [B&C] [D&G]					C
			Line symmetry in the trace of (x,y) (S) Minimum keeps repeating [D&G]		[D&G]	A D F
Symmetry	y is symmetric to x [J&T]					B
			(x,y) repeats the same path for each N units that x moves [D&G]		[D&G]	A C
			Term period (S) Frequency of turning points [B&C]		[B&C]	A D F
Periodicity	y has a repetitive path [J&T]	[J&T]	Term period (S) length of domain that (x,y) takes to make a 'revolution' [J&T]		[J&T]	B C D
	Oscillatory striker (S) (x,y) has repetitive roots [J&A]					B I
Other properties			The graph passes through (0,0) (S) x, y and (x,y) is at zero at the same time [J&T]	[J&T]		C
			Infinity (S) y leaves the screen before x does [J&T]		[J&T]	F

Codes used in table 2.3:

(A) Bring terms which motivated discussion about its meaning

(B) Verify algebraic and Cartesian representation while transforming graphs

(C) Examine Variant and invariant properties by the transformations of graphs

(D) Distinguish two or more functions

(E) Make sense of results obtained from transformations which are counter-examples of their previous beliefs

Table 2.3

Connections built while working in FP in each property

Property	Synthesis between FP and PK	Pairs of students	Motivation
Turning points	Point where the graph changes direction (S) turning point	[B&C]	C
	Turning point with curvature determines a parabola (S) Top of a curve	[J&T]	C
	Term turning point and a point of maximum or minimum (S) 'point where a graph stops increasing and starts decreasing'	[J&A]	A
	Term turning points (S) Maximum or minimum of parabolas and sines	[D&G]	A C
	Value of independent coefficient in a quadratic equation (S) value of turning point	[J&T]	B
Constant function	Absence of x at equation (S) 'y is independent of x'	[J&T]	B
	Absence of x at equation (S) y is constant in a horizontal straight line	[D&G]	A B
	Horizontal straight line (S) term 'constant function' (S) 'step between increasing and decreasing functions'	[J&T]	D
	Term constant function (S) horizontal straight line because 'y does not vary'	[J&A]	A
Monotonicity	Term 'increasing' (S) 'after a graph crosses the x-axis, y is positive' (S) sign of linear coefficient (S) Direction of straight line	[D&G]	C
	Term 'increasing' (S) 'y is positive after [x is] zero'	[J&T]	A
	Term 'increasing' (S) direction of straight line	[J&A]	A C
Derivative	Inclination of straight lines (S) 'ratio between y-intercept and x-intercept'	[J&A] [D&G]	C
	Inclination of straight lines (S) 'ratio between absolute values of x and y'	[B&C]	C
Second derivative	Curvature of a parabola (S) term 'irregular proportion'	[B&C]	D
	Curvature of a parabola (S) term 'proportionality and pattern in the variation	[J&A] [J&T]	A C
	Curvature of a parabola (S) Distance between two symmetrical points	[J&A] [D&G]	D
	Curvature of a parabola (S) Variation in the absolute value (overcame)	[J&T]	C
Range	Term 'range' (S) realised the difference between amplitude and range	[B&C] [J&T]	A C
	Realise the invalidity of the link positive angular coefficient of parabolas and positive range (NL)	[B&C] [J&T]	B
	Positive range (S) sign of angular coefficient (NL)	[D&G]	A
	Term 'range' (S) interval between roots (S) range is related to y	[B&C]	A
	Bounded range (S) maximum for graphs without turning points	[D&G]	C E
	Terms maximum and minimum (S) 'maximum depends only on y	[D&G]	C D
	Terms maximum and minimum (S) 'maximum depends only on y	[J&A]	A C
Symmetry	Term 'symmetry' (S) Line symmetry on the y-axis (S) $f(x)=f(-x)$	[J&T]	A
	Term 'symmetry' (S) Line symmetry on the y-axis (S) $f(x)=f(-x)$	[B&C]	C D
	Term 'symmetry' (S) Line symmetry on the y-axis (S) $f(x)=f(-x)$	[J&A]	E
	Term 'symmetry' (S) Specify line of symmetry out of the axes	[J&T] [D&G]	C
	Term 'symmetry' (S) Specify line of symmetry out of the axes	[B&C]	E
Periodicity	Term 'period' (S) Length of the interval after which a trajectory repeated	[J&T]	A C
	Term 'period' (S) Invariant among special points of graph	[B&C]	A
	Term 'period' (S) frequency of turning points	[B&C]	A
	Term 'periodic' (S) Graph with repetitive trace (S) Roots repeat	[J&A] [D&G]	A
Other findings	Recognise graph from its equation	[D&G]	B

Codes used in table 2.4:

Activity:
(A) Match the strikers and the graphs of FP;
(B) Explain the results obtained from transformations of graphs which generate examples and counter-examples
(C) Answer direct questions on how one characteristic in graphs can be corresponded in strikers (or vice-versa) after matching the strikers and graphs
(D) Compare strikers in DynaGraph which represented counter-examples of their own beliefs

Colours of prints:
Red — Facts happened in the final interview
Black - Facts happened in the research environment

Link:
NL - Naive links
GP - Generalisation of perceptions while proceeding with the connections

Table 2.4

Connections between properties in strikers and graphs built in the final interview

				Bridges		Link		
Properties	DG Parallel	Connected both	Graphs and Definitions	DG -> Cart.	Cart. -> DG	NL	GP	Motivation
Turning points	Value of x and y where y changes orientation [J&A]	[J&A]	Coordinates of turning point [J&A]					B
	Point where striker changes orientation [J&A]	[J&A]	Point where graph stops growing to starts decreasing or vice-versa [J&A]					A
	'point where y meets x' [B&C]	[B&C]	Term 'turning point'[B&C]			NL		A
	Point where striker changes orientation [B&C]	[B&C]	Term 'turning point' (S) Point where the graph changes direction [B&C]	[B&C]			GP	A
	'point where y meets x' [D&G]	[D&G]	Term 'turning point' [D&G]			NL		C
	Value of x and y where y changes orientation [D&G]	[D&G]	Coordinates of turning point [D&G]					C
	Bound of motion of y (S) Point where striker changes orientation [D&G]	[D&G]	Term 'turning point' (S) Extreme values of a graph [D&G]	[D&G]			GP	A
	Striker changes orientation [J&T]	[J&T]	Turning point with curve determines a parabola [J&T]	[J&T]		NL		A
	Bound of the motion of y [J&T]	[J&T]	Bound of y in graph [J&T]					C
Constant functions	Striker is motionless (S) y does not move no matter what x does [J&A]	[J&A]	Term 'constant function' (S) Horizontal straight line [J&A]	[J&A]				A
	y is motionless [B&C]	[B&C]	y does not change (S) Horizontal straight line [B&C]	[B&C]				A
	Striker is motionless (S) only x varies [D&G]	[D&G]	Straight line parallel to the x-axis [D&G]	[D&G]				A
Monotonicity	When x is positive, y is positive [J&A]	[J&A]	Term 'increasing' (S) Direction of straight line [J&A]		[J&A]			A
	'y follows x' and not for linear and quadratic functions [B&C]	[B&C]	Term 'increasing' (S) Domain where the graph is increasing [B&C]	[B&C]			GP	A C
	'y follows x' and 'y does not follow x' [D&G]	[D&G]	Direction of straight line formed by (x,y) (s) term 'increasing' [D&G]					A
	Domain where 'y follows x' or not [D&G]	[D&G]	Direction of any graph [D&G]				GP	C
	'y follows x' for straight line [J&T]	[J&T]	Term 'increasing' (S) inclination of straight line [J&T]					A
	'y follows x' for any function [J&T]	[J&T]	Term 'progressive' (S) Inclination of any graph [J&T]	[J&T]			GP	A C
Derivative	Speed of the striker [J&A]	[J&A]	Angle formed by straight line and the x-axis [J&A]					C D
	'Ratio between variations of x and y' [B&C]	[B&C]	Slope of graphs [B&C]	[B&C]			GP	A
	'Ratio between absolute values of x and y' [D&G]	[D&G]	Inclination of graph [D&G]		[D&G]			A
	Strikers with same speed [D&G]	[D&G]	Ratio between x-intercept and y-intercept [D&G]					B
Second Derivative	Compare speeds of different strikers [J&A]	[J&A]	Compare curvature of sines or parabolas [J&A]					A
	Constant or variable speeds of strikers [D&G] [J&T]	[D&G][J&T]	Curve or straight line [D&G] [J&T]				GP	C D
Range	Point where y returns (S) Bound of the motion of y [J&A]	[J&A]	Extreme values in graphs [J&A]		[J&A]			B
	Bounded and boundless 'place where striker moves' [B&C]	[B&C]	Extreme values in graphs [B&C]					C
	Distinguished 'place where striker moves' from 'length of the place where a striker moves' [J&T] [B&C]	[J&T][B&C]	Distinguished amplitude from range [J&T] [B&C]					B
	Bounded and boundless 'place where striker moves' [D&G]	[D&G]	Extreme values in graphs [D&G]					B
	Side of the axes in DG Parallel [D&G]	[D&G]	Term 'Quadrants' [D&G]			NL		A
	Constant striker stays only on one point [D&G]	[D&G]	'y occupies only one point' [D&G]				GP	B
	Striker gets out of the screen [J&T]	[J&T]	Boundless range [J&T]					C
Symmetry	Striker stays in symmetric values [J&A]	[J&A]	y=x and y=-x are symmetric graphs [J&A]			NL		B
	'Steps that a striker moves to one side are the same as it does to the other side' [B&C]	[B&C]	Line symmetry on the y-axis for parabolas [B&C]		[B&C]			C
	Striker makes the same motion [J&T]							A
	Striker makes the same motion [D&G]	[D&G]	Line symmetry out of the y-axis [D&G]		[D&G]			C
Periodicity	'The striker passes at zero each four steps x moves' [J&A]	[J&A]	Periodicity of roots [J&A]		[J&A]			A
	Oscillation between 'y is negative' and 'y is positive' [J&A]							A
	'How many units x moves while y goes and return' [B&C]	[B&C]	Term 'period' (S) Frequency of special points [B&C]		[B&C]			C
	Striker repeats the same interval [D&G]	[D&G]	Values that y reaches in graph [D&G]		[D&G]			B
	'y repeats the same path for N steps x moves' [D&G]	[D&G]	Periodicity of the trace of the graph (S) Oscillatory graph [D&G]		[D&G]			B
	Periodic motion of the strikers [J&T]	[J&T]	Repetitive trace of graph [J&T]					A
	Distance which x moves from zero up to the point where y starts repeating the motion [J&T]	[J&T]	Term 'period' [J&T]		[J&T]			B

3 Associations and obstacles

Codes used in table 3.1

Origins:
[A] - As a legitimate way of recognising the property among the functions used in the research environment;
[B] - Corresponding the invariants or variants while transforming graphs
[C] - Linking perceptions from different microworlds
[D] - Amplifier of students' associations - Lenses
[E] - Perceptions with same object or using same adjective (language)
[F] - Transforming a property into a polarised rule
[G] - Looking at a property in a pointwise way, mainly the properties of variations
[H] - Students' generalisation between different properties emphasised in different families
[I] - School emphasis for the property to a restricted number of families of functions for which the association is valid and absence of counter-example
[J] - School emphasis on special points

Revised by:
[T] - trying to generalise or analyse the property in a counter-example of the associations
[P] - Discussion generated by students' different points of view
[M] - Matching the strikers with the graphs
[MS] - Matching strikers from both DG microworlds
[GC] - Generating counter-examples by transforming graphs
[K] - Bringing ideas from previous knowledge to explain the association
[E] - Continuing exploring the function in the microworld
[Q] - Answering direct questions in the final interview
[NCM] - The students did not revise the association but neither did they pass through critical moments
[WCM] - The students did not revise the association despite passing through critical moments

Leading to:
[G] - Generalise the perception among ...
[R] - realise that ...
[S] - strengthen the perception by changing to ...
[D] - Distinguished ....

Microworlds:
FI - Final interview
PT - Pre-test
DGs - DG Par. and DG Cart.

Table 3.1
Associations between different perceptions

Properties	Association	Micro world	Origins	Revision	Leading to
Turning points	[B&C] Turning point (A) 'y meets x'	DG Par	D E J	[M]	[S] 'point where striker changes orientation'
	[B&C] Turning point (A) point where 'parabola' changes direction	PT DG par	I	[NCM]	
	[D&G] Turning point (A) 'y meets x'	DG Par in FI	D E	[WCM]	
	[J&T] Turning point (A) 'top of a parabola'	PT FP	I J	[NCM]	
Constant functions	[J&A] A dot (A) absence of x in the equation (A) x does not vary	FP	E	[GC]	Absence of x in the equation (S) horizontal straight line
	[B&C] y is motionless (A) (x,y) is motionless	DG Cart	C E	[MS]	Explain horizontal straight line because y does not vary
	[D&G] A dot (A) absence of x in the equation (A) x does not vary	PT FP	E	[GC]	Absence of x in the equation (S) y is independent of x
	[D&G] y is motionless (A) (x,y) is motionless	DG Cart	C E	[P]	Explain horizontal straight line because y does not vary
Monotonicity	[J&A] 'y follows x' (A) 'y and x is at the same side'	DG Par	F G	[T] [P]	[G] to quadratic functions
	[J&A] Term increasing (A) 'function which reaches positive values'	PT	F I	[M] in FI	Term increasing (S) orientation of the strikers
	[J&A] Term increasing (A) direction of straight line	FP	I	[WCM]	
	[B&C] Term increasing (A) 'the straight line is positive to the left side after a value'	DG Cart	D F I	[M][Q] in FI	[G] term increasing to all the functions
	[B&C] Term increasing (A) direction of straight line	PT FP	I	[M][Q] in FI	[R] direction in all of the graphs
	[D&G] Term increasing (A) 'a' is positive (A) after x-intercept, y is positive (A) direction of straight line	FP	D F I	[WCM]	
	[D&G] Term increasing (A) 'a' is positive (A) positive curvature of parabola	PT FP DG Cart	D F H	[WCM]	
	[J&T] Term increasing (A) direction of straight line (A) 'after x is zero, y is positive'	PT FP DGs	E F I	[M][E][Q] in FI	Term 'increasing' (S) orientation of the striker of linear functions.
	[J&T] 'x and y moves to the same side' (A) 'y is over x' (A) 'x and y have the same speed'	DG	A	[T]	[G] for non-linear functions
Derivative	[J&A] Speed of strikers (A) 'ratio between absolute values of x and y'	DG Par	A G	[T] [P]	[D] constant and variable ratio of variations of x and y
	[J&A] Inclination of straight line (A) ratio between x-intercept and y-intercept	FP	B	[NCM]	
	[J&A] Inclination of straight line (A) ratio between absolute values of x and y	FP	B	[T][Q] in FI	Inclination of straight line (S) ratio between variations of x and y
	[B&C] Linear coefficient (A) ratio between absolute values of x and y	FP	B	[M] in FI	
	[B&C] Comparing speed (A) Which striker is ahead	DG Par.	G	[T] [K]	Enable them to compare speed of different strikers
	[B&C] Speed of strikers (A) 'ratio between absolute values of x and y'	DGs	A	[T] [M] in FI	[S] for 'ratio between variations of x and y'
	[D&G] y has same speed of x (A) abs(y)=abs(x)	DG Par	A	[T]	[G] Speed for all the functions
	[D&G] Inclination of straight line (A) ratio between absolute values of x and y	FP	A G	[GC] in FI	Inclination of graph (S) speed of strikers
	[D&G] Comparing speed (A) Which striker is ahead	DG Par	A G	[T] [K]	[R] the invalidity of the association
	[D&G] Ratio between angles formed by straight line with the axes (A) linear coefficient	FP DG Cart	A	[WCM]	
	[D&G] Inclination of straight lines (A) 'ratio between x-intercept and y-intercept'	FP	B	[WCM]	
	[D&G] Speed of strikers (A) 'ratio between absolute values of x and y'	DG Par	A	[T]	[S] ratio between variations of x and y
	[J&T] Comparing speed (A) Which striker is ahead	DGs	D G	[T]	Enable them to compare speed of different strikers
	[J&T] Speed of strikers (A) 'ratio between absolute values of x and y'	DG Par	A C	[T]	[S] changing to a variational ratio
	[J&T] Inclination of straight line (A) 'ratio between absolute values of x and y'	FP	B	[M]	[S] changing to a variational ratio
	[J&T] Inclination of straight line (A) imaginary angle in DG Parallel	DG Par	D E	[T]	Inclination of straight line (S) Speed of striker
Second Derivative	[J&A] Curvature (A) 'distance between roots'	FP	C G	[GC]	Developed a way to measure curvatures
	[B&C] Variable speed (A) y overtakes x	DG Par.	G	[T] [K]	Enable them to compare curvature of different strikers
	[B&C] Curvature (A) 'distance between two symmetrical points'	FP	D G	[GC]	Developed a way to measure curvatures
	[D&G] Curvature (A) height of y for a fixed x	FP	A D G	[T][Q] in FI	Inclination of graphs (S) speed of strikers
	[D&G] Curvature (A) 'distance between two symmetrical points'	FP	D G	[GC]	[R] the invalidity of measuring curvature in this way
	[D&G] Variable speed (A) y overtakes x	DG	D G	[T]	[R] That the change of speed depended on the point
	[J&T] Curvature (A) 'distance between two symmetrical points'	DG	D G	[GC]	[R] invariance of curvature under vertical translation
	[J&T] Comparing curvature (A) comparing speed	DG Par	A C	[T][Q] in FI	[D] fixed and variable 'ratio between absolute value' (S) curve or straight line
	[J&T] Variation of absolute value of x (A) curvature	FP	B	[GC][T]	Curvature (S) pattern of variation
Range	[J&A] Extreme values (A) turning points	PT DG Cart FP	I J	[GC]	Extreme values (S) y of the turning point
	[J&A] Maximum (A) positive value	FP	G J	[WCM]	
	[B&C] Term 'range' (A) bounded range	FP	I	[P]	[G] the term 'range' for boundless functions
	[B&C] Positive or negative range (A) sign of angular coefficient	FP	D E F	[GC]	Sign of angular coefficient (S) positive curvature of parabola
	[D&G] Extreme values (A) turning point and concavity	FP	D I J	[E]	[G] boundary of range for all the functions
	[D&G] Positive or negative range (A) sign of angular coefficient	FP	E F	[WCM]	
	[J&T] Positive or negative range (A) sign of angular coefficient	FP	E F	[GC]	Sign of angular coefficient (S) positive curvature of parabola
Symmetry	[J&A] Line symmetry (A) symmetric numbers	FP	I	[GC] in FI	[R] $f(x)=f(-x)$ in strikers
	[B&C] Line symmetry (A) line symmetry on the y-axis (A) $f(x) = f(-x)$	PT FP	A I	[GC]	Identify line of symmetry and [R] the invalidity of $f(x)=f(-x)$ on line symmetry about a vertical line different from the y-axis
	[J&T] Line symmetry (A) $f(x) = f(-x)$ (A) symmetric numbers	PT FP	I	[GC]	[D] line symmetry from symmetric numbers
Periodicity	[J&A] Period (A) distance between two roots	DG Cart FP	A G J	[NCM]	
	[J&A] 'y oscillates' (A) y is independent of x	PT DG Par	D F	[E]	[D] oscillatory from independent
	[J&A] Periodic (A) oscillatory	DG Par.	A I	[M] in FI	[R] the periodicity in the frequency of roots
	[B&C] Periodic (A) repetition of 'special points'	DG Cart FP	J	[GC]	[R] its invariance when calculated among special points
	[B&C] 'y oscillates' (A) 'y is independent of x'	DG Par	F	[WCM]	
	[D&G] Periodic (A) repetition of the graph (A) repetition of the roots	FP	A G I J	[WCM]	
	[D&G] 'y oscillates' (A) y is independent of x	DG Par	F	[E]	[R] period units x moves to y start to repeat
	[D&G] Periodic (A) oscillatory graph	FP DG Par	A I	[E]	[R] periodicity by frequency that the trace repeats
	[J&T] Period (A) interval which of one cycle takes (A) interval between roots	FP	A D G I J	[GC] [E]	[E] period as interval of roots to distance bet. roots
	[J&T] Period (A) the trace of a cycle	FP	A I	[GC]	[D] period and trace
	[J&T] Periodic (A) oscillatory graph	DG Par	A I	[NCM]	
	[J&T] 'y oscillates' (A) 'y is independent of x'	DG Par	F	[E]	[D] oscillatory from independent
Other properties	[J&A] Straight lines (A) strikers which moves only in one orientation	DGs	A I J	[NCM]	
	[J&A] [B&C] Parabola (A) curve with turning point	DG Cart	I J	[NCM]	
	[D&G] Linear function (A) graph pass through origin	FP DGs	D G I J	[WCM]	
	[D&G] Parabola (A) curve with turning point	DGs	D I J	[NCM]	
	[J&T] Parabola (A) curve with turning point	FP DG Par.	D I J	[NCM]	
	[J&T] Coefficient 'a' (A) positive curvature	FP	D H	[P]	[R] the invalidity of generalising 'a' as the same coefficient in different families of functions

Codes used in table 3.2:

A - Emphasise a property in only one family of functions
B - Equations as essence of functions
C - Pointwise view
D - Polarisation of knowledge
E - Language used

Table 3.2
Summary of moments when obstacles were clearly observed

Cause of barrier	Pairs of Students	Description of the barrier making them unable to:	Origins	Transposition
Monotonicity as direction of straight line and the term 'increasing'	[B&C] [J&T]	generalise the variational perception of monotonicity developed in DG microworlds to all the functions	A	A strong generalisation of 'y follows x' to strikers of parabola and sines allowed them to overcome the barrier in the final interview only
Monotonicity by term increasing by rule 'after x-intercept, y is positive'	[J&A]	link the term increasing to the variational perception developed in FP when generalised to parabola	A D	It was not transposed
Monotonicity meaning sign of linear coefficient	[D&G]	analyse monotonicity by real meaning transforming it in 'a' is positive	A B D	It was not transposed
Slope as 'angle between straight line and the x-axis'	[J&T]	analyse 'ratio between variations of x and y' in striker corresponding to parabola after recognising it should be a parabola arguing that curve has no angles	A	It was transposed only in the final interview when they constructed the distinction between constant and variable derivative in strikers and graphs
Recognition of the family to which a function belongs	[D&G] [J&T]	search for other characteristics different from those they studied in the family after recognising to which family the function belongs	A	Tendency of this pair of students
Recognition of equation of striker	[J&T]	compare similarities or differences between strikers after recognising the equation to which it corresponds	B	Tendency of this pair of students
Polarisation while analysing range	[D&G] [J&T] [B&C]	recognise the similarities between range of $y=0.25x^2$ and $y=0.25x^2-8$	D	All these pairs adopted the analysis of range by bounded or boundless range
Polarisation while analysing domain	[J&A]	recognising similarity between y=x and y=-x while looking at monotonicity. Then, they argued that y=-x is a similar function to $y=-0.25x^2$	D	It was not transposed
Polarisation while analysing domain	All the pairs	generalise the perception 'y follows x' to strikers of sines, leading them to perceive sine as 'y is independent of x'	D	It was transposed for all the pairs apart from [B&C]
Not specifying the variable or subject while speaking or writing	All the pairs	connect the properties in a proper way, leading them to connect by the characterisation	E	The nature of describing/guessing activity demanded of the students more precision in order to allow their colleague to guess each description. Especially in the interaction with DG Parallel, a completely new representation where a common language had not previous been built, the students were led to specify the variables.
Separation from variation for strikers and pointwise for functions	[D&G] [J&T]	Connecting properties observed by motion from properties observed pointwisely	C	Transposed in the development of the activities

4 Exploration of DG microworlds

Codes used in tables 4.1 and 4.2:

A - The students separated the variables x, y and (x,y) they were talking about

B - The students searched for a functional view of a property previously perceived pictorially

C - The students created a new way of analysing the property in graphs.

D - The perceptions stay isolated in DG Parallel until the final interview

E - The students present a pictorial perception

F - The students present variational perception

G - The students maintain a pointwise view (at least until the final interview)

H - The students present polarisation in the perception

I - The students generalise the perception among other kind of functions with global view

Table 4.1

Main aspects of the students' perceptions of each property in DG Parallel

	FP -> DG		DG -> FP	
	John & Tanya	Diana & Gisele	Bernard & Charles	Jane & Anne
Turning points	F	F	D F	D F
Constant functions	A F	A F	A F	A D F
Monotonicity	A F	A F	A F	A F
Derivative	A F (linear F.)	A F (all F.)	A F (all F.) G	A F (all F.)
Second derivative	A	A F	A F	A F
Range	F H	F H	F H	
Symmetry				
Periodicity	A F			

Table 4.2

Main aspects of the students' perceptions of each property in DG Cartesian

	FP -> DG		DG -> FP	
	John & Tanya	Diana & Gisele	Bernard & Charles	Jane & Anne
Turning points	A C	A C	E	E
Constant functions	A C	A B	A C	A E F
Monotonicity	A E F	A E F	E	A B E F
Derivative	A C F	A B D E F	A C (lin.) F	A C F
Second derivative	A B F	A D F	A C E F	A B C E F
Range	F H I	A F H I	F H I	E
Symmetry		B F	B	
Periodicity	B	A B F	B F	B F